2025/FYUG/EVEN/SEM/
MATDSC–253/220

FYUG Even Semester Exam., 2025

MATHEMATICS

(4th Semester)

Course No. : MATDSC–253

(Linear Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following : $2 \times 2 = 4$

(a) Let $V(F)$ be a vector space and $\hat{0}$ be the zero element of V. Then show that $\alpha x = \hat{0} \Rightarrow \alpha = 0$ or $x = \hat{0}$ $\forall\, \alpha \in F$ and $\forall\, x \in V$.

(b) Justify whether the set

$$W = \{(x, y, 2) \mid x, y \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3(\mathbb{R})$.

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(c) Consider a subset $S = \{(1, 4), (0, 3)\}$ of $\mathbb{R}^2(\mathbb{R})$. Show that $(2, 3) \in L(S)$.

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Let V be the set of all pairs (x, y) of real numbers. Define

$$(x_1, y_1) + (x_2 + y_2) = (x_1 + x_2, y_1 + y_2)$$
$$\alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$$

Show that with these operations V is a vector space over the field of real numbers $\mathbb{R}$. 6

(b) Define linearly dependent set and linearly independent set. Show that the set

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$$

is linearly dependent. 1+1+2=4

(c) Show that the union of two subspaces of a vector space is also a subspace if and only if one of them contains the other. 4

(d) If S and T are two subsets of a vector space $V(F)$, prove that—

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$;

(ii) $L(S \cup T) = L(S) + L(T)$;

(iii) $L(L(S)) = L(S)$. 2+2+2=6

UNIT—II

3. Answer any *two* of the following : 2×2=4

(a) Define basis of vector space with a suitable example.

(b) Define dimension of a finite dimensional vector space. What is the dimension of $P_3(\mathbb{R})$?

(c) Define quotient space.

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that every finitely generated vector space has a basis. 4

(b) If W_1 and W_2 are two subspaces of a finite dimensional vector space $V(F)$, then prove that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 6

(c) Show that every linearly independent subset of a finite dimensional vector space is a basis or can be extended to form a basis. 5

(d) If W be a subspace of a finite dimensional vector space $V(F)$, then prove that

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$

5

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) Is $T:\mathbb{R}^2(\mathbb{R})\to\mathbb{R}^3(\mathbb{R})$ defined as $T(x, y) = (x+3, 2y, x+y)$ a linear transformation? Justify your answer.

(b) Let $T:\mathbb{R}^2\to\mathbb{R}^2$ is linear $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 4)$. What is $T(2, 3)$?

(c) Define nullity and rank of a linear transformation.

6. Answer *either* [(a) and (b)] or [(c) and (d)] :

(a) If V is a vector space and $T: V\to V$ is a linear operator, then prove that the following are equivalent : 4

(i) Range $(T)\cap\ker(T)=\{\hat{0}\}$

(ii) $T(T(x)) = 0 \Rightarrow T(x) = 0$

(b) Show that the mapping $T:\mathbb{R}^2(\mathbb{R})\to\mathbb{R}^3(\mathbb{R})$ defined as $T(x, y) = (x+y, x-y, y)$ is a linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$. Find Null space, Range space, Nullity and Rank of T. 6

(c) State and prove Sylvester's law of nullity. 1+4=5

(d) Let T be linear operator on $\mathbb{R}^3$ defined as $T(x, y, z) = (2y+z, x-4y, 3x)$. Compute the matrix T relative to the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 5

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) Define eigenvalue of a linear operator. Give an example.

(b) Prove that eigenvalues of unitary matrix are of unit modulus.

(c) Write the characteristic polynomial of

$$A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

8. Answer *either* [(a) and (b)] or [(c) and (d)] :

(a) Show that eigenvalues of

$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}$$

are ± 1 and the corresponding eigenvectors are

$$\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \text{ and } \begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}$$ 5

(b) State and prove Cayley-Hamilton theorem. 5

(c) Let T be a linear operator on a finite dimensional vector space V over a field F. Prove that $\alpha \in F$ is an eigenvalue of T if and only if $T - \alpha I$ is singular. 5

(d) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence find A^{-1}. 5

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) Define an inner product space.

(b) In an inner product space $V(F)$, prove that $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

(c) Define orthogonality of two vectors. Also define orthogonal set.

10. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) State and prove Cauchy-Schwarz inequality. 1+5=6

(b) If $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal subset of an inner product space V, then prove that for any $v \in V$, the vector

$$v - \sum_{i=1}^{n} (v, v_i) v_i$$

is perpendicular to each v_i. 4

(c) State and prove Bessel's inequality in an inner product space. 5

(d) Prove that for all $x, y, \in V$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\{\|x\|^2 + \|y\|^2\}$$

where V is an inner product space. 5

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