FYUG Even Semester Exam., 2025

MATHEMATICS

(4th Semester)

Course No. : MATDSC-251

(Abstract Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : $2\times2=4$

(a) Define a group. Give a suitable example.

(b) Write the Cayley table for the set $G = \{0, 1, 2, 3\}$ with respect to usual multiplication. Is it a group?

(c) Define order of an element of a group. Find the order of 2 in $\mathbb{Z}_4$.

2. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 10

(a) Show that the set $GL(2, \mathbb{R})$ of 2×2 matrices with real entries and non-zero determinant is a group w.r.t. matrix multiplication. 5

(b) Show that a non-empty subset H of a group G is a subgroup of G if and only if ab^{-1} is in H whenever a, b are in H. 5

(c) Prove the uniqueness of identity and uniqueness of inverses in a group. 2+2=4

(d) Let

$$G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$$

Show that G is a group w.r.t. multiplication. 6

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Let $U(14)$ be the group of positive integers less than 14 and relatively prime to 14, w.r.t. multiplication modulo 14. Find <11> i.e., the subgroup generated by 11.

(b) Define centralizer of an element of a group.

(c) Let $G = < a >$ be a cyclic group of order 25. Justify if a^5 is a generator of G.

4. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 10

(a) Show that the center of a group is a subgroup of the group. 5

(b) Let $G = < a >$ be a cyclic group of order n. Then prove that $G = < a^k >$ if and only if $\gcd(k, n) = 1$. 5

(c) Justify if the center of a group is always Abelian. Justify if the centralizer of an element of a group is always Abelian. 5

(d) Show that every subgroup of a cyclic group is cyclic. 5

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Define symmetric group S_n.

(b) What is the order of the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{pmatrix}?$$

(c) If H is a subgroup of a group G, show that $aH = H$ if and only if $a \in H$.

6. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 10

(a) Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. Express the permutation $(1\,2)(4\,5)(1\,5\,3)(2\,4)$ as a product of disjoint cycles. 4+1=5

(b) Let H be a subgroup of a group G. Show that any two right cosets of H in G are either disjoint or identical. 5

(c) Show that disjoint cycles commute. Show that S_{2025} is non-Abelian by giving example of two cycles that do not commute. 4+1=5

(d) Consider the subgroup

$$H = \{0, \pm 5, \pm 10, \pm 15, \cdots\}$$

of the group $\mathbb{Z}$ of integers w.r.t. addition. Check if the cosets $2025 + H$ and $5202 + H$ are same or different. Find all the cosets of H in $\mathbb{Z}$. Justify your answer. 2+3=5

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Let G be a group of order 43. Find all the subgroups of G. Justify your answer.

(b) Justify if the center of a group is a normal subgroup of the group.

(c) Write all the elements of the factor group $\mathbb{Z}/4\mathbb{Z}$.

8. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 10

(a) State and prove Lagrange's theorem. 1+5=6

(b) Show that a subgroup H of a group G is normal in G if and only if $xHx^{-1} \subseteq H$ for all $x \in G$. 4

(c) Suppose that G is a group of order pq where p and q are primes. Prove that every proper subgroup of G is cyclic. 4

(d) Let G be a group and H be a normal subgroup of G. Show that the set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = (ab)H$. 6

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) Give example, with justification, of a non-commutative ring with unity.

(b) Give example, with justification, of an integral domain that is not a field.

(c) Justify if an integral domain can have characteristic 2025.

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) Show that a non-empty subset S of a ring R is a subring if and only if S is closed under subtraction and multiplication. 5

(b) Prove that every non-zero finite integral domain is a field. 5

(c) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. Show that $\mathbb{Z}[\sqrt{2}]$ is a ring under ordinary addition and multiplication of real numbers. 5

(d) An element x of a ring R is called nilpotent if $x^n = 0$ for some positive integer n. Show that the nilpotent elements of a commutative ring form a sub-ring of the ring. 5

★★★