FYUG Odd Semester Exam., 2025

MATHEMATICS

(5th Semester)

Course No. : MTMDSC–302T

(Topology)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : $2\times2=4$

(a) Define a metric and a metric space, and also give an example of each.

(b) Describe open spheres in a discrete metric space.

(c) In a metric space (X, d), justify that the empty set ϕ is both open and closed.

(Turn Over)

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Let $\mathbf{C}$ be the set of all complex numbers and $d : \mathbf{C} \times \mathbf{C} \to \mathbb{R}$ is defined as $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$ $\forall x, y \in \mathbf{C}$. Prove that d is a bounded metric on $\mathbf{C}$. 10

(b) Prove that a subset A of a metric space X is closed if and only if A^c is an open set. 5

(c) Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ is a function. Prove that d is a metric on X if and only if d satisfies the two conditions

(i) $d(x, y) = 0 \Leftrightarrow x = y \;\; \forall x, y \in X$

(ii) $d(x, y) \le d(x, z) + d(y, z) \;\; \forall x, y, z \in X$ 5

(d) In a metric space, prove that any finite set is a closed set. 5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) In the real line $\mathbb{R}$ with its usual metric, describe the interior of the set $\mathbb{Q}$ of all rational numbers and of the set $A = (0, 2]$.

(Continued)

(b) Define a dense set in a metric space and give an example of it.

(c) Prove that a sequence $\langle x_n \rangle$ in a metric space (X, d) converges to a point $x \in X$ if and only if the sequence $\langle d(x_n, x) \rangle$ converges to zero.

10

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Let (X, d) be a metric space and $A \subseteq X$. Prove that the boundary of A is given by $b(A) = \overline{A} \cap \overline{A^c}$, where $\overline{A}$ and $\overline{A^c}$ are closures of A and A^c. 5

(b) Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two Cauchy sequences in a metric space (X, d). Prove that the sequence $\langle d(x_n, y_n) \rangle$ is convergent. 5

(c) Let (X, d) be a metric space and $A \subseteq X$. Prove that A is dense in X if and only if the only open set disjoint from A is the empty set ϕ. 5

(d) If a convergent sequence in a metric space has infinitely many distinct points, then prove that the limit of the sequence is also a limit point of the range set of the sequence. 5

(Turn Over)

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Define a complete metric space and give an example of it.

(b) Prove or disprove :
The set of all rational numbers $\mathbb{Q}$ is a complete metric space with respect to the metric d defined by $d(x, y) = |x - y|$ $\forall\, x, y \in \mathbb{Q}$.

(c) Let (X, d) be a metric space and $x_0 \in X$ be a fixed point. Define a function $f : X \to \mathbb{R}$ by $f(x) = d(x, x_0)$ $\forall x \in X$. Examine if f is continuous.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) State and prove Cantor's intersection theorem. 5

(b) Let X and Y be two metric spaces and $f : X \to Y$ be a mapping. Prove that f is continuous if and only if $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$. 5

(*Continued*)

(c) Prove that a mapping $f : X \to Y$ from a metric space X to a metric space Y is continuous at a point $x_0 \in X$ if and only if for every sequence $\langle x_n \rangle$ in X converging to x_0, the sequence $\langle f(x_n) \rangle$ in Y converges to $f(x_0)$. 5

(d) Prove that $\mathbb{R}^3$ is a complete metric space with respect to its usual metric. 5

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Give an example, with justification, of a proper non-empty subset of a topological space which is both open and closed.

(b) Describe closed sets in a co-finite topological space (X, T).

(c) Prove or disprove :
$\{G \subseteq \mathbb{R} : \mathbb{R} - G \text{ is infinite}\} \cup \{\mathbb{R}\}$ is a topology on $\mathbb{R}$.

(*Turn Over*)

8. Answer *either* [(*a*) and (*b*)] *or* [(*c*) and (*d*)] : 10

(*a*) Prove that

$$T = \{G \subseteq \mathbb{R} : \mathbb{R} - G \text{ is countable}\} \cup \{\mathbb{R}\}$$

is a topology on $\mathbb{R}$. 5

(*b*) Prove that a subset A of a topological space X is closed if and only if A contains its boundary $b(A)$. 5

(*c*) Let (X, T) be a topological space and $A \subseteq X$. Prove that the following are equivalent :

(*i*) Int (A) is the union of all open subsets of A.

(*ii*) Int (A) is the largest open subsets of A. 5

(*d*) (*i*) Give an example of a topological space other than discrete space and indiscrete space, in which open sets are exactly same as closed sets. 3

(*ii*) Show that the collection of all open intervals in $\mathbb{R}$ is not a topological space. 2

UNIT—V

Answer any *two* of the following questions :

2×2=4

(*a*) Define a metrizable space. Is every topological space metrizable? Justify your answer.

(*b*) Define continuity at a point of a function $f : X \to Y$, where X and Y are two topological spaces.

(*c*) Let T_1 and T_2 denote the discrete and usual topology on $\mathbb{R}$ and $f : (\mathbb{R}, T_1) \to (\mathbb{R}, T_2)$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. Examine if f is continuous.

0. Answer *either* [(*a*) and (*b*)] *or* [(*c*) and (*d*)] : 10

(*a*) Let X and Y be two topological spaces and $f : X \to Y$ is a bijective mapping. Prove that f is a homeomorphism if and only if f and f^{-1} are both open mapping. 5

(b) Let X be an infinite set and T be the co-finite topology on X. Prove that (X, T) is not metrizable.

(c) Prove that the open interval $(-1, 1)$ and $\mathbb{R}$ are homeomorphic with respect to usual topology on $\mathbb{R}$ and the corresponding relative topology on $(-1, 1)$. 5

(d) Let (X, T) be a metrizable space and x, y are two distinct points in X. Prove that there exist two disjoint open sets G and H such that $x \in G$ and $y \in H$. 5

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