2025/FYUG/ODD/SEM/ MTMDSC–201T/067

FYUG Odd Semester Exam., 2025

MATHEMATICS

(3rd Semester)

Course No. : MTMDSC–201T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

   (a) If $S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, then show that $\inf S = 0$.

   (b) If x and y are real numbers with $x < y$, then show that there exists an irrational number z such that $x < z < y$.

   (c) Let $a, b \in \mathbb{R}$. Then prove that
   $$a < b + \frac{1}{n} \ \forall n \in \mathbb{N} \text{ iff } a \le b$$

(Turn Over)

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) State and prove nested interval property. 10

(b) (i) If A and B are bounded subsets of real numbers, then prove that $A \cup B$ and $A \cap B$ are also bounded. 5

(ii) Give an example each of a bounded set which contains its— 3

(1) supremum but does not contain its infimum;

(2) infimum but does not contain its supremum. 1+1=2

(c) Show that there is no rational number r such that $r^2 = 8$. 3

(d) (i) Let A and B be bounded non-empty subsets of $\mathbb{R}$, and let

$$A + B = \{a + b : a \in A, \ b \in B\}$$

Prove that

$$\sup(A+B) = \sup A + \sup B$$
$$\inf(A+B) = \inf A + \inf B$$ 5

(ii) Let $J_n = \left(0, \frac{1}{n}\right)$, $n \in \mathbb{N}$. Show that

$$\bigcap_{n=1}^{\infty} J_n = \emptyset$$ 2

(Continued)

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Show that $\underset{n \to \infty}{\text{Lt}} \frac{1}{1+n^2} = 0$.

(b) Define a bounded sequence in $\mathbb{R}$. Give an example of a sequence in $\mathbb{R}$ which is not bounded above and not bounded below.

(c) Show that the sequence $\langle x_n \rangle$, where

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \ \forall \, n \in \mathbb{N}$$

is monotonic increasing.

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) (i) Prove that a convergent sequence in $\mathbb{R}$ is bounded. 3

(ii) Using squeeze theorem, show that

$$\underset{n \to \infty}{\text{Lt}} \left(\frac{\sin n}{n}\right) = 0$$ 2

(b) Show that the sequence $\langle x_n \rangle$, defined by the recursion formula $x_{n+1} = \sqrt{3x_n}$, $x_1 = 1$ is convergent. Also find the limit. 3+2=5

(c) (i) Show that the sequence $\langle n \rangle$ is divergent. 2

(Turn Over)

(ii) Show that the sequence $\langle x_n \rangle$, where

$$x_n = \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n}$$

is convergent.

(d) If $\underset{n \to \infty}{\text{Lt}}\ x_n = l$, then show that

$$\underset{n \to \infty}{\text{Lt}}\ |x_n| = |l|$$

Is the converse true? Justify. 2+2=4

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Define a Cauchy sequence with an example.

(b) Using divergence criteria, show that the sequence

$$\langle 1 - (-1)^{n+1} \rangle$$

is divergent.

(c) Give an example of properly divergent sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ with $y_n \neq 0$ for all $n \in \mathbb{N}$ such that $\left\langle \frac{x_n}{y_n} \right\rangle$ is convergent.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) State and prove Cauchy convergence criterion for real sequences. 6

(b) Show that the sequence $\langle x_n \rangle$, where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not convergent. 4

(c) (i) State monotone subsequence theorem. Using this, show that a bounded sequence of real numbers has a convergent subsequence. 1+2=3

(ii) Show that the sequence $\langle (-1)^n / n \rangle$ is a Cauchy sequence. 3

(d) Let $X = \langle x_n \rangle$ be a real sequence. Then X is said to be contractive if there exists a constant c, $0 < c < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq c\,|x_{n+1} - x_n| \ \forall\, n \in \mathbb{N}$$

Prove that X is convergent if X is a contractive sequence. 4

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Define a cluster point of a set. Find the cluster point of the set

$$\left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$$

(Turn Over)

(b) Using squeeze theorem, show that

$$\underset{x \to 0}{\text{Lt}}\; x \sin \frac{1}{x} = 0$$

(c) Show that $\underset{x \to 0}{\text{Lt}} \frac{1}{x^2}$ does not exist in $\mathbb{R}$.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Let $f : A \to \mathbb{R}$ and c be a cluster point of A. Then the following are equivalent:

(i) $\underset{x \to c}{\text{Lt}}\, f(x) = l$, l is a finite real number.

(ii) For every sequence $\langle x_n \rangle$ in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $\langle f(x_n) \rangle$ converges to l.

(b) Prove that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 \ , & x \text{ is irrational} \\ -1 \ , & x \text{ is rational} \end{cases}$$

is nowhere continuous.

(c) Using ε-δ definition of continuity, prove that the function $f(x) = x^2 \;\forall\, x \in \mathbb{R}$ is continuous of any point in $\mathbb{R}$.

(d) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(r) = 0 \;\forall\, r \in \mathbb{Q}$. Show that $f(x) = 0 \;\forall\, x \in \mathbb{R}$.

(Continued)

(ii) Using sequential criterion of continuity, show that the function $f(x) = c \;\forall\, x \in \mathbb{R}$, c is a constant, is continuous at every point in $\mathbb{R}$. 2

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) State boundedness theorem for a continuous function on a closed bounded interval. Is the theorem true for bounded open interval?

(b) State Caratheodory theorem.

(c) State Lagrange mean value theorem.

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 10

(a) If f is differentiable on $I = [a, b]$ and if $k \in \mathbb{R}$ is a number between $f'(a)$ and $f'(b)$, then show that there exists at least one point $c \in (a, b)$ such that $f'(c) = k$. 5

(b) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then show that f has an absolute maximum and an absolute minimum on I. 5

(Turn Over)

(c) Let f be continuous on the closed bounded interval $I = [a, b]$ and differentiable on (a, b) such that

$$f'(x) = 0 \ \forall \ x \in (a, b)$$

Then prove that f is constant on I.

(d) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I.

(i) If $k \in \mathbb{R}$ with $\inf f(I) \le k \le \sup f(I)$, then prove that there exists a number $c \in I$ such that $f(c) = k$.

(ii) Prove that $f(I)$ is a closed bounded interval.

(iii) Give an example to show that if I is a bounded open interval then $f(I)$ may not be an open interval.

3+3+1=

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