Unit-4 Afjal Sir notes

Orthogonal Curvillinear Coordinates

Defination:

The curvillinear are the common name of differents pairs of coordinates other than cartesian coordinates.

Let, (x, y, z) be the cartesian coordinates of any point p and (u_1, u_2, u_3) are new coordinates which are related to (x, y, z) with the relation.

$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1\ u_2\ u_3)$$
$$z = z(u_1\ u_2\ u_3)$$

Solving the above equation we can write,

Let, $u_1\ u_2\ u_3$ in terms of (x, y, z) represented by

$$u_1 = u_1(x, y, z)$$
$$u_2 = u_2(x, y, z)$$
$$u_3 = u_3(x, y, z)$$

The above set of equations defines trans-formation of coordinates. hence, the functions are single value and have continuous derivatives. As a result the corresponding between (x, y, z) and $(u_1 u_2 u_3)$ is unit, These coordinates $u_1 u_2 u_3$ are called curvillinear coordinates.

Some common examples of curvillinear Coordinates are —

1. Spherical polar coordinate.
2. Cylindrical coordinate.
3. Paraboidal coordinate.

* Orthogonal curvilinear coordinates

An any curvillinear coordinates system there are coordinates & surfaces corresponding to $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1, c_2, c_3 are const. The interaction of each pair of surface results a curve line called coordinate line.

A system of curvillinear coordinates u_1, u_2, u_3 is said to be orthogonal if the tangent to the coordinates line are mutually perpendicular.

* Unit vector of curvillinear coordinate system

The unit vector along the tangents to the coordinate lines at a point are called unit vectors in the curvillinear coordinates. They are denoted by

$$u_1 \rightarrow \hat{e}_1$$
$$u_2 \rightarrow \hat{e}_2$$
$$u_3 \rightarrow \hat{e}_3$$

If the position vectors of a points whose cartesian coordinates are (x, y, z) and the corresponding curvillinear coordinates are (u_1, u_2, u_3)

The rate of variation of $\vec{r}$ along the tangent of first quadrant is given by $\frac{\partial \vec{r}}{\partial u_1}$. This is a vector quantity along tangent to u_1 line and hence, it is called tanget vector along this line is

$$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left|\frac{\partial \vec{r}}{\partial u_1}\right|}$$

Similarly,

$$\hat{e}_2 = \frac{\frac{\partial \vec{r}}{\partial u_2}}{\left|\frac{\partial \vec{r}}{\partial u_2}\right|} \quad \text{and} \quad \hat{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left|\frac{\partial \vec{r}}{\partial u_3}\right|}$$

if h_1, h_2, h_3 be the corresponding modulus of tangent vector:

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

Then,

$$\frac{\partial \vec{r}}{\partial u_1} = h_1 \hat{e}_1, \quad \frac{\partial \vec{r}}{\partial u_2} = h_2 \hat{e}_2 \text{ and } \frac{\partial \vec{r}}{\partial u_3} = h_3 \hat{e}_3$$

now,

$$\vec{r} = h_1 \hat{e}_1 + h_2 \hat{e}_2 + h_3 \hat{e}_3$$

× **Expression for arc length in curvillinear coordinates**

In curtesian coordinate system the length element is represented by

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$
$$ds^2 = d\vec{r} \cdot d\vec{r} \quad \text{—} ①$$

Here

$$\vec{r} = \vec{r}(u_1\ u_2\ u_3)$$

$$\therefore d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$
$$= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \quad \left(\because \frac{\partial \vec{r}}{\partial u_1} = \hat{e}_1 h_1\right)$$

from ①

$$ds^2 = d\vec{r} \cdot d\vec{r}$$
$$= (h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3) \cdot (h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3)$$
$$= h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Therefore, length element.

$$ds = \sqrt{h_1^2 du_1^2 + h_2 du_2^2 + h_3^2 du_3^2}$$

We know,

The distance between any two points in the curvillinear coordinate system is called the line arc element, which is given by

$$ds_i = h_i du_i$$

In this case there are 3 possibles surface area element in the orthogonal curvillinear system which are given by

$$dA_{ij} = ds_i\, ds_j \qquad i, j = 1, 2, 3$$
$$= h_i h_j \cdot du_i\, du_j \qquad i \neq j$$

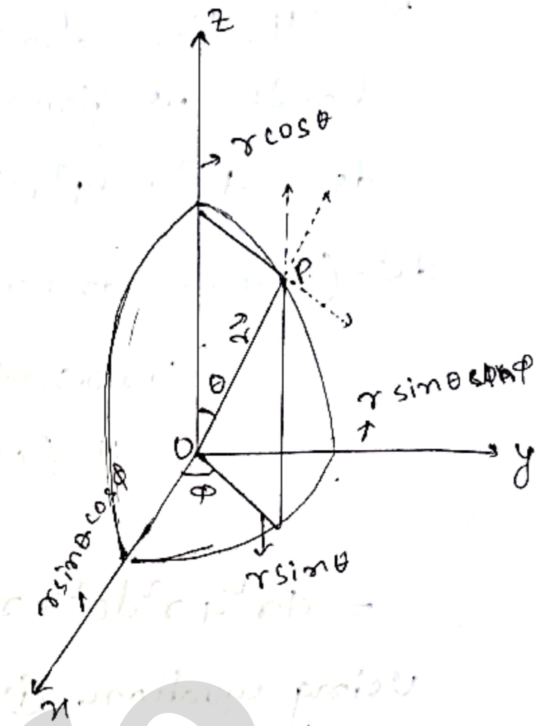
* Volume Element in orthogonal curvillinear System

The volume element in orthogonal curvillinear is given by

$$d\tau = ds_i \cdot ds_j \cdot ds_k$$
$$= h_i\, h_j\, h_k \cdot du_i\, du_j\, du_k \qquad [\, i, j, k = 1, 2, 3, \; \hat{i} \neq j \neq k \,]$$

* Spherical polar coordinates as sperical curvillinear coordinates.

Let p be a point in space. Such that it curvillinear points are $(u_1, u_2 u_3)$ and curtesian points are (x, y, z) then, the spherical polar coordinates can be specified by $u_1 = r$, $u_2 = \theta$, and $u_3 = \phi$



Here,

$\vec{r} = OP$, the distance of the point from origin.
θ is the angle between of $\vec{OP}$ and z axis
ϕ is the angle between xz plane and opz plane.

From the figure the transformation equation, represented by.

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Now,

$$\therefore dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi \quad \text{—} (1)$$

$$= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$$

$$\therefore dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi \quad \text{—} (2)$$

$$= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$$

$$\therefore dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta + \frac{\partial z}{\partial \phi}d\phi$$

$$= \cos\theta\, dr - r\sin\theta\, d\theta + 0$$

$$= \cos\theta\, dr - r\sin\theta\, d\theta \quad \text{—} (3)$$

So, The line element in spherical polar coor-dinate is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$ds^2 = (\sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi)^2 + (\sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi)^2 + (\cos\theta\, dr - r\sin\theta\, d\theta)^2$$

$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \quad \text{---(4)}$$

Using equation (1), (2) and (3)

but the line element in curvillinear coordi-nates is given by

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

In spherical polar coordinate the length element is given by

$$ds^2 = h_1^2 dr^2 + h_2^2 d\theta^2 + h_3^2 d\phi^2 \quad \text{---(5)}$$

comparing (4) and (5)

$h_1 = 1$,

$h_2 = r$,

$h_3 = r\sin\theta$.

So, the volume element in spherical polar coordinates is given by

$$d\tau = h_1 h_2 h_3\, du_1 du_2 du_3$$

$$d\tau = 1 \cdot r \cdot (r\sin\theta)\, dr\, d\theta\, d\phi$$

$$\boxed{d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi}$$

Unit vector of spherical polar coordinates

We have, position vector is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

Now, The unit vectors along the curvilinear coordinates are –

$$\therefore \hat{e}_1 = \hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$$

$$\hat{e}_r = \frac{1}{1}\left\{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right\}$$

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\therefore \hat{e}_2 = \hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\hat{e}_\theta = \frac{1}{r}\left\{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}\right\}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \hat{k}$$

$$\therefore \hat{e}_3 = \hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

$$\hat{e}_\phi = \frac{1}{r\sin\theta}\left\{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0\right\}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \;///$$

Now,

$$\hat{e}_\theta \cdot \hat{e}_\phi = \left\{\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \hat{k}\right\} \cdot \left\{-\hat{i}\sin\phi + \cos\phi\,\hat{j}\right\}$$

$$= -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi$$

$$= 0.$$

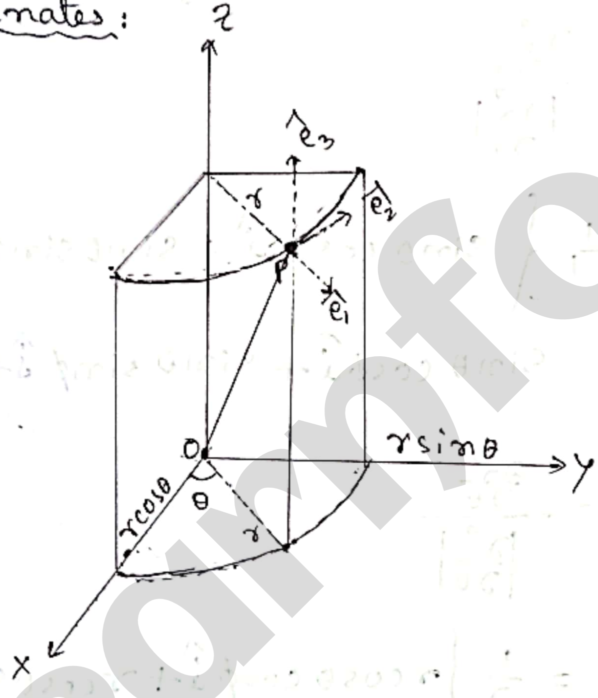
$\because \vec{e}_r \cdot \hat{e}_\theta = 0,$

and,

$\vec{e}_r \cdot \hat{e}_\phi = 0,$

So, the unit vectors are mutually perpendicular to each other. i.e; They are orthogonal

* Cylindrical coordinate as spherical curvilinear coordinates:



Let P be a point on the space such that its curvillinear coordinates are (u_1, u_2, u_3) and cartesian coordinates are (x, y, z). Let the position of P in xy plane be θ, where the polar coordinates of the point is (r, θ). So, the cylindrical coordinate of point P is given by

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = z$$

The transformation equation of the coordinate can be obtain from the adjacent figure.

$$x = r\cos\theta,$$
$$y = r\sin\theta$$
$$z = z,$$

$\therefore dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial z} dz$

$= \cos\theta dr - r\sin\theta d\theta + 0$

$\therefore dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial z} dz$

$= \sin\theta dr + r\cos\theta d\theta + 0$

$\therefore dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial z} dz$

$= dz$

So, the line element can be expressed as

$ds^2 = dx^2 + dy^2 + dz^2$

$= (\cos\theta dr - r\sin\theta d\theta)^2 + (\sin\theta dr + r\cos\theta d\theta)^2 + dz^2$

$= dr^2 + r^2 d\theta^2 + dz^2 \quad \text{--- (1)}$

but the line element in curvilinear coordinates is given by

$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$

$= h_1^2 dr^2 + h_2^2 d\theta^2 + h_3^2 dz^2 \quad \text{--- (2)}$

$\therefore$ Comparing equation (1) and (2) we get

$h_1 = 1,$

$h_2 = r,$

$h_3 = 1$

So, the volume element in cylindrical coordinate is given by

$d\tau = h_1 h_2 h_3 du_1 du_2 du_3$

$d\tau = 1 . r . 1 \; dr d\theta dz$

$\boxed{d\tau = r dr d\theta dz}$

* **Unit vector in Cylindrical coordinates system**

We have, the position vector in cylindrical coordinate is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$

Now,

The unit vectors along the curvilinear coordinates are –

$$\therefore \hat{e}_1 = \hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$$

$$\hat{e}_r = \frac{1}{1}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\therefore \hat{e}_2 = \hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\hat{e}_\theta = \frac{1}{r}\left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right)$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\therefore \hat{e}_3 = \hat{e}_z = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\hat{e}_z = \frac{1}{1}(\hat{k})$$

$$\hat{e}_z = \hat{k}$$

Now,

$$\therefore \hat{e}_r \cdot \hat{e}_\theta = \left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)\cdot\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$$= -\sin\theta\cos\theta + \sin\theta\cos\theta$$

$$= 0.$$

$$\therefore \hat{e}_\theta \cdot \hat{e}_z = 0.$$

$$\therefore \hat{e}_r \cdot \hat{e}_z = 0,$$

So, the unit vector are mutually perpendicular to each other that is they are orthogonal.

Basic

gradient → $\vec{\nabla}\cdot\phi$ → vector. (a nebla operation)

divergent → $\vec{\nabla}\cdot\vec{A}$ → scalar

curl → $\vec{\nabla}\times\vec{A}$ → vector

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* Gradient in Orthogonal curvilinear coordinates

let us consider a scalar function ϕ, such that

$$\phi = \phi(u_1, u_2 u_3)$$

then the gradient of the scalar function is given by,

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial u_1}\vec{\nabla}u_1 + \frac{\partial\phi}{\partial u_2}\vec{\nabla}u_2 + \frac{\partial\phi}{\partial u_3}\vec{\nabla}u_3 \quad \text{—(1)}$$

But from the orthogonal curvilinear coordinat system, we have,

$$\vec{\nabla}u_1 = \frac{\hat{e}_1}{h_1}, \quad \vec{\nabla}u_2 = \frac{\hat{e}_2}{h_2}, \quad \vec{\nabla}u_3 = \frac{\hat{e}_3}{h_3}$$

from —(1), we get,

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial u_1}\frac{\hat{e}_1}{h_1} + \frac{\partial\phi}{\partial u_2}\frac{\hat{e}_2}{h_2} + \frac{\partial\phi}{\partial u_3}\frac{\hat{e}_3}{h_3}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\hat{e}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\phi}{\partial u_3}$$

$$\Rightarrow \vec{\nabla}\phi = \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)\phi$$

$$\Rightarrow \vec{\nabla} = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}$$

(This required for value of nebla($\vec{\nabla}$))

So,
① In spherical polar coordinates, if,

$u_1 = r$, $u_2 = \theta$, $u_3 = \phi$

also,

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

∴ In spherical polar coordinate (nebla)

$$\vec{\nabla} = \hat{e}_1 \frac{\partial}{\partial r} + \frac{\hat{e}_2}{r}\frac{\partial}{\partial \theta} + \frac{\hat{e}_3}{r\sin\theta}\frac{\partial}{\partial \phi}$$

gradient in spherical polar coordinate.

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{\phi}$$

② In cylindrical polar coordinate if

$u_1 = r$, $u_2 = \theta$, $u_3 = z$

and

$h_1 = 1$, $h_2 = r$, $h_3 = 1$

In ~~spherical~~ cylindrical polar coordinate (nebla)

$$\vec{\nabla} = \hat{e}_1 \frac{\partial}{\partial r} + \frac{\hat{e}_2}{r}\frac{\partial}{\partial \theta} + \hat{e}_3\frac{\partial}{\partial z}$$

gradient in cylindrical polar coordinate

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z}$$

* Expression for divergence in orthogonal curvillinear coordinate:

Let $\vec{A}$ be a vector quantity

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

also an orthogonal curvillinear coordinate.

$$\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = h_2 h_3 (\vec{\nabla} u_2 \times \vec{\nabla} u_3)$$

$$\therefore \vec{A} = A_1 h_2 h_3 (\vec{\nabla} u_2 \times \vec{\nabla} u_3) + A_2 h_3 h_1 (\vec{\nabla} u_3 \times \vec{\nabla} u_1) + A_3 h_1 h_2 (\vec{\nabla} u_1 \times \vec{\nabla} u_2)$$

$$\therefore \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\left[A_1 h_2 h_3 (\vec{\nabla} u_2 \times \vec{\nabla} u_3) + A_2 h_3 h_1 (\vec{\nabla} u_3 \times \vec{\nabla} u_1) + A_3 h_1 h_2 (\vec{\nabla} u_1 \times \vec{\nabla} u_2)\right]$$

Now, from vector Indentity,

$$\vec{\nabla}\cdot(\phi\vec{A}) = \phi\vec{\nabla}\cdot\vec{A} + (\vec{\nabla}\phi)\cdot\vec{A}$$

so.

$$\vec{\nabla}\cdot\left\{A_1 h_2 h_3 (\vec{\nabla} u_2 \times \vec{\nabla} u_3)\right\} = A_1 h_2 h_3\, \vec{\nabla}\cdot(\vec{\nabla} u_2 \times \vec{\nabla} u_3) + \vec{\nabla}(A_1 h_2 h_3)\cdot(\vec{\nabla} u_2 \times \vec{\nabla} u_3) \quad \text{—(1)}$$

Now,

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}(\vec{\nabla}\times\vec{A}) - \vec{A}(\vec{\nabla}\times\vec{B})$$

$\left[\because \text{curl/grad}\,\phi = 0 \quad \therefore \vec{\nabla}\times\vec{\nabla}\phi = 0\right]$

so,

$$\vec{\nabla}\cdot(\vec{\nabla} u_2 \times \vec{\nabla} u_3) = \vec{\nabla} u_3 (\vec{\nabla}\times\vec{\nabla} u_2) - \vec{\nabla} u_2 (\vec{\nabla}\times\vec{\nabla} u_3)$$

$$= 0 - 0$$

$$= 0$$

$\left[\because \text{since } \vec{\nabla}\times\vec{\nabla}\phi = 0.\right]$

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So from equation (1) we get.

$$\vec{\nabla}\cdot\left\{A_1 h_2 h_3 (\vec{\nabla}u_2 \times \vec{\nabla}u_3)\right\} = \vec{\nabla}(A_1 h_2 h_3)\cdot(\vec{\nabla}u_2 \times \vec{\nabla}u_3)$$

$$= \vec{\nabla}(A_1 h_2 h_3)\cdot\left(\frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3}\right)$$

$$= \vec{\nabla}(A_1 h_2 h_3)\cdot \frac{\hat{e}_1}{h_2 h_3} \quad \text{--- (2)}$$

Now,

$A_1 h_2 h_3$ is function of $u_1 u_2 u_3$. so.

$$\vec{\nabla}(A_1 h_2 h_3) = \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\vec{\nabla}u_1 + \frac{\partial}{\partial u_2}(A_1 h_2 h_3)\vec{\nabla}u_2 + \frac{\partial}{\partial u_3}(A_1 h_2 h_3)\vec{\nabla}u_3$$

$$= \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial u_2}(A_1 h_2 h_3)\frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(A_1 h_2 h_3)\frac{\hat{e}_3}{h_3}$$

So, from equation (2) we can write,

$$\vec{\nabla}\cdot\left\{A_1 h_2 h_3 (\vec{\nabla}u_2 \times \vec{\nabla}u_3)\right\} = \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\frac{1}{h_1 h_2 h_3}$$

$$\therefore \vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial u_1}(A_1 h_2 h_3)\frac{1}{h_1 h_2 h_3} + \frac{\partial}{\partial u_2}(A_2 h_3 h_1)\frac{1}{h_1 h_2 h_3} + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\frac{1}{h_1 h_2 h_3}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$

$$= \boxed{\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(A_r r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}}$$

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Note

① In spherical polar coordinate

$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$

and

$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$

So, divergence in an spherical polar coordinate

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_1 r^2\sin\theta) + \frac{\partial}{\partial\theta}(A_2 r\sin\theta) + \frac{\partial}{\partial\phi}(A_3 r)\right]$$

② In cylindrical polar coordinate

$u_1 = r, \quad u_2 = \theta, \quad u_3 = z$

and.

$h_1 = 1, \quad h_2 = r; \quad h_3 = 1.$

So, divergence in a cylindrical polar coordinate

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(A_1 r) + \frac{\partial}{\partial\theta}(A_2) + \frac{\partial}{\partial z}(\cancel{z} A_3 r)\right]$$

(*) Expression for curl in orthogonal curvilinear coordinate

Let us consider a vector function of $\vec{A}$. Such that,

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

$$\therefore \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(A_1\hat{e}_1) + \vec{\nabla}\times(A_2\hat{e}_2) + \vec{\nabla}\times(A_3\hat{e}_3)$$

$$= \vec{\nabla}\times(A_1 h_1 \nabla u_1) + \vec{\nabla}\times(A_2 h_2 \nabla u_2) + \vec{\nabla}\times(A_3 h_3 \vec{\nabla} u_3) \quad \text{--- (1)}$$

Now,

using vector In Indentity. we have,

$$\vec{\nabla}\times(\phi\vec{A}) = \phi(\vec{\nabla}\times\vec{A}) + (\vec{\nabla}\phi)\times\vec{A}$$

$$\therefore \vec{\nabla} \times (A_1 h_1 \vec{\nabla} u_1) = A_1 h_1 (\vec{\nabla} \times \vec{\nabla} u_1) + \vec{\nabla}(A_1 h_1) \times \vec{\nabla} u_1$$

$$= \vec{\nabla}(A_1 h_1) \times \vec{\nabla} u_1 \qquad \left\{ \because \vec{\nabla} \times \vec{\nabla} u_1 = 0 \right.$$

$$= \vec{\nabla}(A_1 h_1) \times \frac{\hat{e}_1}{h_1} \quad \text{--- (2)}$$

Since, $A_1 h_1$ is function of $u_1 u_2 u_3$.

$$\therefore \vec{\nabla}(A_1 h_1) = \frac{\partial}{\partial u_1}(A_1 h_1) \vec{\nabla} u_1 + \frac{\partial}{\partial u_2}(A_1 h_1) \vec{\nabla} u_2 + \frac{\partial}{\partial u_3}(A_1 h_1) \vec{\nabla} u_3$$

$$= \frac{\partial}{\partial u_1}(A_1 h_1) \frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial u_2}(A_1 h_1) \frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(A_1 h_1) \frac{\hat{e}_3}{h_3}$$

So, equation (2) becomes.

$$\vec{\nabla} \times (A_1 h_1 \vec{\nabla} u_1) = \left\{ \frac{\partial}{\partial u_1}(A_1 h_1) \frac{\hat{e}_1}{h_1} + \frac{\partial}{\partial u_2}(A_1 h_1) \frac{\hat{e}_2}{h_2} + \frac{\partial}{\partial u_3}(A_1 h_1) \frac{\hat{e}_3}{h_3} \right\} \times \frac{\hat{e}_1}{h_1}$$

$$= \frac{\partial}{\partial u_2}(A_1 h_1) \frac{(-\hat{e}_3)}{h_1 h_2} + \frac{\partial}{\partial u_3}(A_1 h_1) \frac{\hat{e}_2}{h_3 h_1}$$

$$\cancel{= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_3}(A_1 h_1) h_2 \hat{e}_2 - \frac{\partial}{\partial u_2}(\right.}$$

$$= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_3}(A_1 h_1) h_2 \hat{e}_2 - \frac{\partial}{\partial u_2}(A_1 h_1) h_3 \hat{e}_3 \right.$$

So, from equation (i) we get.

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\left\{ \frac{\partial}{\partial u_3}(A_1 h_1) h_2 \hat{e}_2 - \frac{\partial}{\partial u_2}(A_1 h_1) h_3 \hat{e}_3 \right\} + \left\{ \frac{\partial}{\partial u_1}(A_2 h_2) h_3 \hat{e}_3 - \frac{\partial}{\partial u_3}(A_2 h_2) h_1 \hat{e}_1 \right\} + \left\{ \frac{\partial}{\partial u_2}(A_3 h_3) h_1 \hat{e}_1 - \frac{\partial}{\partial u_1}(A_3 h_3) h_2 \hat{e}_2 \right\} \right]$$

$$\Rightarrow \vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Note

① In spherical polar coordinate

$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$

and

$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta$

So, curl in spherical polar coordinate.

$$\vec{\nabla} \times \vec{A} = \frac{1}{\cancel{h_1 h_2 h_3}\ r^2 \sin\theta} \begin{vmatrix} \hat{e}_1 & r\hat{e}_2 & r\sin\theta\, \hat{e}_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & A_2 r & A_3 r\sin\theta \end{vmatrix}$$

② In cylindrical polar coordinate

$u_1 = r, \quad u_2 = \theta, \quad u_3 = z$

and

$h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

So, curl in ~~spherical~~ cylindrical polar coordinate

$$\vec{\nabla} \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{e}_1 & r\hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & A_2 r & A_3 \end{vmatrix}$$

*. Expression for Laplacian in orthogonal curvilinear coordinate

We have the gradient in orthogonal curvi-llinear coordinate is given by

$$\vec{\nabla} = \left(\frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \right)$$

$\therefore$ Laplacian

$$\nabla^2 = (\vec{\nabla})^2$$

$$= \vec{\nabla}\cdot\vec{\nabla}$$

$$= \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)\cdot \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)$$

$$= \left(\frac{1}{h_1}\frac{\partial}{\partial u_1}\right)^2 + \left(\frac{1}{h_2}\frac{\partial}{\partial u_2}\right)^2 + \left(\frac{1}{h_3}\frac{\partial}{\partial u_3}\right)^2$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$