Unit-5

Beta & Gamma function

Beta f^n:- The beta f^n or Eular integration of first ~~time~~ Kind is defined as.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ where } m,n>0$$

Properties of B funct^n:-

1. The B funct^n is symmetrical in m & n.

i.e, $B(m,n) = B(n,m)$.

Proof:- from the defination of B funct^n.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad \text{---} (1)$$

If we make changing in the variable

$$x = 1-y$$

then eq^n (1) we have,

$$B(m,n) = \int_1^0 (1-y)^{m-1}y^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$= B(n,m).$$

2. $B(m,n) = \dfrac{m-1}{m+n-1}B(m-1,n) = \dfrac{n-1}{m+n-1}B(m,n-1)$

Proof :

from the defination of βf^n we have,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Intigrating by Parts we get,

$$\beta(m,n) = \left[(1-x)^{n-1}\frac{x^m}{m}\right]_0^1 + \int_0^1 (n-1)(1-x)^{n-2}\frac{x^m}{m}\,dx$$

The first term of the right hand side is 0.

$$\therefore \beta(m,n) = \frac{n-1}{m}\int_0^1 (1-x)^{n-2}x^m\,dx \quad \text{—(1)}$$

Now, $x^m = x^{m-1} - x^{m-1}(1-x)$

Substituting this in eqn we get,

$$\therefore \beta(m,n) = \frac{n-1}{m}\int_0^1 \{x^{m-1} - x^{m-1}(1-x)\}\cdot(1-x)^{n-2}\,dx$$

$$\Rightarrow \beta(m,n) = \frac{n-1}{m}\int_0^1 x^{m-1}(1-x)^{n-2} - x^{m-1}(1-x)\cdot(1-x)^{n-2}\,dx$$

$$\Rightarrow \beta(m,n) = \frac{n-1}{m}\int_0^1 x^{m-1}(1-x)^{(n-1)-1} - \frac{n-1}{m}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\Rightarrow \beta(m,n) = \frac{n-1}{m}\beta(m,n-1) - \frac{n-1}{m}\beta(m,n).$$

$$\Rightarrow \beta(m,n)\left[1+\frac{n-1}{m}\right] = \frac{n-1}{m}\beta(m,n-1)$$

$$\Rightarrow \beta(m,n)\left[\frac{m+n-1}{m}\right] = \frac{n-1}{m}\beta(m,n-1)$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{n-1}{m+n-1}\beta(m,n-1)}$$

interchanging m and n we get,

$$\beta(n,m) = \frac{m-1}{m+n-1}\beta(n,m-1)$$

$$\Rightarrow \boxed{\beta(m,n) = \frac{m-1}{m+n-1}\beta(m-1,n)}$$

3. $\beta(m,n) = \beta(n,m) = \dfrac{\lfloor m-1 \lfloor n-1}{\lfloor m+n-1}$; m, n are integers

Proof :

we have, $\beta(m,n) = \dfrac{n-1}{m+n-1}\beta(m,n-1)$.

Applying this relation successively we get,

$$\beta(m,n) = \frac{n-1}{m+n-1}\cdot\frac{n-2}{m+n-2}\cdot\frac{n-3}{m+n-3}\cdots\frac{1}{m+1}\beta(m,1).$$

if m and n are both integers then eqn (1) becomes.

$$\beta(m,n) = \frac{(n-1)(n-2)\ldots 3,2,1}{(m+1)(m+2)\ldots(m+n-1)} \times \frac{(m-1)(m-2)\ldots 3,2,1}{m(m-1)(m-2)\ldots 2,1}\beta(1,1)$$

$$= \frac{\lfloor n-1 \lfloor m-1}{\lfloor m+n-1}\beta(1,1) \quad \text{—(2)}$$

Now,

$$B(1,1) = \int_0^1 x^{1-1}(1-x)^{1-1}\,dx$$

$$= \int_0^1 dx = [x]_0^1 = 1.$$

So, eqn ② becomes,

$$B(m,n) = \frac{\lfloor n-1 \lfloor m-1}{\lfloor m+n-1}$$

4. $B(m,n) = \dfrac{1}{2^{2m-1}} \, B\left(\tfrac{1}{2}, m\right)$

<u>Proof</u>:

from the defination of B f^n

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Putting $m=n$

we get, $B(m,m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx$

$$= \int_0^1 \{x(1-x)\}^{m-1}\,dx$$

$$= \int_0^1 (x-x^2)^{m-1}\,dx$$

$$= \int_0^1 \left(\frac{1}{4} - \frac{1}{4} + \frac{2x}{2} - x^2\right)^{m-1} dx$$

$$= \int_0^1 \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{m-1} dx.$$

$$= 2\int_0^{1/2} \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{m-1} dx$$

substitution, $\frac{1}{2} - x = \frac{\sqrt{y}}{2}$ If $x=\frac{1}{2}$ then $y=0$

$\Rightarrow dx = \frac{-1}{4\sqrt{y}}\,dy$ $x=0$ then $y=1$

So eqn ① becomes,

$$B(m,n) = 2\int_1^0 \left(\frac{1}{4} - \frac{y}{4}\right)^{m-1}\left(\frac{-1}{4\sqrt{y}}\right) dy$$

$$= \frac{\cancel{2}}{2^{2(m-1)}} \int_0^1 (1-y)^{m-1}\left(\frac{1}{\cancel{4}_2}\, y^{-1/2}\right) dy$$

$$= \frac{1}{2^{2(m-1)}} \int_0^1 y^{-1/2}(1-y)^{m-1}\,dy$$

$$= \frac{1}{2^{2(m-1)}} \int_0^1 y^{\frac{1}{2}-1}(1-y)^{m-1}\,dy$$

$$= \frac{1}{2^{2(m-1)}}\, B\left(\frac{1}{2}, m\right).$$

Different forms of B $funct^n$:

1. from the defination of B f^n we have,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{—①}$$

Putting $x = \frac{y}{1+y}$ then, $1-x = 1 - \frac{y}{1+y}$ If $x=1$ then $y=\infty$

again, $dx = \frac{dy}{(1+y)^2}$ If $x=0$ then, $y=0$

Putting this value in eqⁿ ① we get,

$$B(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \frac{d}{(1+y)^2}$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m-1+n-1+2}} dx$$

or, $$\boxed{B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy} \quad \text{—②}$$

2. If we substitute $y = \frac{\alpha}{\beta} x$ in eqⁿ ② where α/β is const.

$$B(m,n) = \int_0^\infty \frac{(\alpha/\beta\, x)^{m-1}}{(1+\frac{\alpha}{\beta}x)^{m+n}} \left(\frac{\alpha}{\beta}\right) dx$$

$$= \int_0^\infty \frac{(\alpha/\beta)^m x^{m-1}}{(\beta+\alpha x)^{m+n} \beta^{-(m+n)}} dx$$

$$= \int_0^\infty \frac{(\alpha/\beta)^m \cdot \beta^{(m+n)}}{(\beta+\alpha x)^{m+n}} x^{m-1} dx$$

$$\Rightarrow B(m,n) = \alpha^m \beta^n \int_0^\infty \frac{x^{m-1}}{(\beta+\alpha x)^{m+n}} dx$$

3. We know,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad \text{—①}$$

If we substitude $x = \sin^2\theta$

then $dx = 2\sin\theta\cos\theta\, d\theta$

So eqⁿ ① ⇒

$$B(m,n) = \int_0^{\pi/2} (\sin\theta)^{2(m-1)} \cdot (\cos\theta)^{2(n-1)}\, 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow \boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta\, d\theta}$$

Γ-fⁿ:- Γ-fⁿ or Eulers integ. of 2nd kind is defined as

$$\Gamma m = \int_0^\infty x^{m-1} e^{-x} dx \,;\; m>0$$

Recursance formula for Γ-fⁿ:- From the defination of Γ-fⁿ we have $\Gamma m = \int_0^\infty x^{m-1} e^{-x} dx$

$$\therefore \Gamma(m+1) = \int_0^\infty x^m e^{-x} dx$$

Integrating by Parts we get,

$$\therefore \Gamma(m+1) = \left[-x^m e^{-x}\right]_0^\infty + \int_0^\infty m x^{m-1} e^{-x} dx$$

The first term of the right hand side vanises for both the limits.

$\therefore \Gamma(m+1) = \int_0^\infty m x^{m-1} e^{-x} dx$

$\Rightarrow \Gamma(m+1) = m \int_0^\infty x^{m-1} e^{-x} dx$

$\Rightarrow \boxed{\Gamma(m+1) = m\,\Gamma(m)}$ — ①

$\Rightarrow \Gamma(m+1) = m(m-1).(m-2) = \cdots 2, 1\, \Gamma(1)$

$\Rightarrow \Gamma(m+1) = \lfloor m \; \Gamma(1)$ — ②

$\therefore$ Now, $\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$

$\therefore \Gamma(1) = \int_0^\infty x^0 e^{-x} dx$

$= \left[-e^{-x}\right]_0^\infty$

$= -(0-1)$

$= 1$

$\therefore$ from eqn ② we get

$\boxed{\Gamma(m+1) = \lfloor m}$

Relation b/w Γ & β - f^n :-

from the defination of Γ f^n we have.

$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$

Substituting $x = ty$ $\forall, t > 0$

we get, $\Gamma(m) = \int_0^\infty (ty)^{m-1} e^{-ty}\, t\, dy$.

Now, Replacing m by $m+n$ & t by $(1+t)$.

we get, $\Gamma(m+n) = (1+t)^{m+n} \int_0^\infty y^{m+n-1} e^{-(1+t)y} dy$

$\Rightarrow \dfrac{\Gamma(m+n)}{(1+t)^{m+n}} = \int_0^\infty y^{m+n-1} e^{-(1+t)y} dy$ — ①

Multiplying eqn ① by t^{m-1}

and integrating lim 0 to ∞ we get,

we get,

$\Gamma(m+n) \int_0^\infty \dfrac{t^{m-1}}{(1+t)^{m+n}} dt = \int_0^\infty \int_0^\infty t^{m-1} y^{m+n-1} e^{-(1+t)y} dy\, dt$

$\Rightarrow \Gamma(m+n)\, \beta(m,n) = \int_0^\infty y^{m+n-1} \left[\int_0^\infty t^{m-1} e^{-(1+t)y} dt\right] dy$

$$= \int_0^\infty y^{n-1} \left[\int_0^\infty (ty)^{m-1} e^{-ty} \, d(ty) \right] e^{-y} dy$$

$$= \int_0^\infty y^{n-1} (\Gamma m) e^{-y} dy$$

$$= \Gamma m \, \Gamma n$$

$$\Rightarrow \Gamma(m+n) \, B(m,n) = \Gamma m \, \Gamma n$$

$$\Rightarrow B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}.$$

∴ Value of $\Gamma\frac{1}{2}$:- from the defination of Γ-f^n we have,

$$\Gamma m = \int_0^\infty x^{m-1} e^{-x} dx$$

$$\therefore \Gamma\tfrac{1}{2} = \int_0^\infty x^{-1/2} e^{-x} dy$$

Substituting $x = y^2$

ie $dx = 2y\,dy$ we get,

$$\Gamma\tfrac{1}{2} = \int_0^\infty y^{-1} e^{-y^2} 2y\,dy$$

$$= 2\int_0^\infty e^{-y^2} dy \quad -①$$

changing the variable.

$$\Gamma\tfrac{1}{2} = 2\int_0^\infty e^{-z^2} dz \quad -②$$

Multiplying eqn ① & ② we get,

$$(\Gamma\tfrac{1}{2})^2 = 4\int_0^\infty \int_0^\infty e^{-(y^2+z^2)} dy\,dz$$

Substituting $y = r\cos\varphi$ & $z = r\sin\varphi$

we get,

$$(\Gamma\tfrac{1}{2})^2 = 4\int_{\varphi=0}^{\pi/2} \left[\int_{r=0}^\infty r e^{-r^2} dr \right] d\varphi.$$

$$= 4\int_{\varphi=0}^{\pi/2} \left[-\tfrac{1}{2} e^{-r^2} \right]_0^\infty d\varphi$$

$$= 4\int_0^{\pi/2} \tfrac{1}{2} \, d\varphi$$

$$= 2[\varphi]_0^{\pi/2}$$

$$= \cancel{2} \, \tfrac{\pi}{\cancel{2}}$$

$$\therefore (\Gamma\tfrac{1}{2})^2 = \pi$$

$$\Rightarrow \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Value of $\Gamma\frac{3}{2}$:

We have $\Gamma(m+1) = m\Gamma m$

$\Rightarrow \cancel{\Gamma\frac{1}{2}+1 = \frac{1}{2}\sqrt{\pi}}$ $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\sqrt{\pi}$

$\Rightarrow \cancel{\Gamma\frac{3}{2} = \sqrt{\pi}/2} \Rightarrow \Gamma\frac{3}{2} = \frac{\sqrt{\pi}}{2}$

Value of $\Gamma 0$:

We have, $\Gamma(m+1) = m\Gamma m$

$\Rightarrow \Gamma m = \frac{\Gamma(m+1)}{m}$

$\Rightarrow \Gamma 0 = \frac{\Gamma 1}{0}$

$\Rightarrow \Gamma 0 = \infty$

Value of $\Gamma(-1)$

We have, $\Gamma(m+1) = m\Gamma m$

$\Rightarrow \Gamma m = \frac{\Gamma(m+1)}{m}$

$\Rightarrow \Gamma(-1) = \frac{\Gamma(-1+1)}{-1} =$

$\Rightarrow \Gamma(-1) = \frac{\infty}{-1} = -\infty$

Q. Show that, $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

~~Sol~~

(Km aile Prove korte lagbe)

we have, $B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$ — ①

also, Relat^n b/w B & Γ f^n.

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \quad \text{— ②}$$

comparing ① and ② we have,

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \quad \text{(Proved)}$$

Q. Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$

Sol from the defination B f^n we have,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Substituting, $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$.

We get, $B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{—①}$$

but the relⁿ b/w β & Γ fⁿ we have,

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{—②}$$

Comparing ① and ② we get,

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{2\Gamma m+n}$$

Putting $2m-1 = p$ and $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$ $\Rightarrow n = \frac{q+1}{2}$

We have,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} \quad \text{(Proved)}$$

Special Case:

1. Substituting $q = 0$ we get,

$$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\Gamma p+1/2 \, \Gamma 1/2}{2\Gamma\frac{p+2}{2}} = \frac{\sqrt{\pi} \, \Gamma\frac{p+1}{2}}{2\Gamma\frac{p+2}{2}}$$

2. Substituting $p = 0$ we get,

$$\int_0^{\pi/2} \cos^q\theta \, d\theta = \frac{\Gamma\frac{1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{q+2}{2}} = \frac{\sqrt{\pi} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{q+2}{2}}$$

Value of $\Gamma\frac{1}{2}$:

we have, $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$

Putting $p = 0$ and $q = 0$ we get,

$$\int_0^{\pi/2} d\theta = \frac{\Gamma\frac{1}{2} \, \Gamma\frac{1}{2}}{2\Gamma 1}$$

$$\Rightarrow [\theta]_0^{\pi/2} = \frac{(\Gamma 1/2)^2}{2} \quad (\because \Gamma 1 = 1)$$

$$\Rightarrow (\Gamma 1/2)^2 = 2\,\frac{\pi}{2}$$

$$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$

Q. Show that $\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$

Sol: from the defination of β-f^n, we have,

$$\beta(m,n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \quad \text{—} ①$$

Putting $y = \frac{1}{x}$ in the second integral of RHS of eqn ① we get,

$$I_2 = \int_1^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_1^{0} \frac{\left(\frac{1}{x}\right)^{m-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(-\frac{1}{x^2}\right)dx = \frac{\left(\frac{1}{x}\right)^{m-1} x^{m+n} x^{-2}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{-m+1+m+n-2}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

changing the variable, we can write

$$I_2 = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy \quad \text{—} ②$$

Putting the value of eqn ② in ① we get,

$$\beta(m,n) = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\Rightarrow \beta(m,n) = \int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}\,dy \quad \text{—} ③$$

again from the ~~eqn~~ relt^n b/w β and Γ-f^n

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad \text{—} ④$$

comparing the eqn ③ and ④ we have,

$$\int_0^1 \frac{y^{m-1}+y^{n-1}}{(1+y)^{m+n}}\,dy = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Q. Show that $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

we can write $\int_0^{\infty} \frac{x^8-x^{14}}{(1+x)^{24}}\,dx$

$$= \int_0^{\infty} \frac{x^8}{(1+x)^{24}}\,dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{9+15}}\,dx$$

$$= \beta(9,15) - \beta(15,9)$$

$$= 0 \qquad \{\beta(m,n) = \beta(n,m)\}$$

Q. Express and Product $2, 5, 8, \ldots (3n-1)$ in term of Γ function.

Sol: We have, $\Gamma(n+1) = n\Gamma(n)$

replacing n by $(n-\frac{1}{3})$ we get,

$\Gamma(n-\frac{1}{3}+1) = (n-\frac{1}{3})\Gamma(n-\frac{1}{3})$

$\Rightarrow \Gamma(n+\frac{2}{3}) = (n-\frac{1}{3})(n-\frac{4}{3})\Gamma(n-\frac{4}{3})$ $\{-\frac{1}{3}-1 = -\frac{4}{3}\}$

$\Rightarrow \Gamma(n+\frac{2}{3}) = \frac{(3n-1)(3n-4)(3n-7)\ldots 8, 5, 2\,\Gamma(\frac{2}{3})}{3^n}$

reverse

$\Rightarrow \frac{3^n\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})} = 2, 5, 8 \ldots (3n-1)$

$\therefore 2, 5, 8, \ldots (3n-1) = \frac{3^n\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})}$

Q. If n is a positive integer Prove that,

$2^n\Gamma(n+\frac{1}{2}) = 1, 3, 5 \ldots (2n-1)\sqrt{\pi}$

Sol: We have, $\Gamma(n+1) = n\Gamma(n)$

replacing n by $(n-\frac{1}{2})$ we get,

$\Gamma(n-\frac{1}{2}+1) = (n-\frac{1}{2})\Gamma(n-\frac{1}{2})$

$\Rightarrow \Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2})\Gamma(n-\frac{3}{2})$

$\Rightarrow \Gamma(n+\frac{1}{2}) = (n-\frac{1}{2})(n-\frac{3}{2}) \ldots \Gamma(\frac{1}{2})$

$\Rightarrow \Gamma(n+\frac{1}{2}) = \frac{(2n-1)(2n-3)(2n-7)\ldots 5, 3, 1\,\Gamma(\frac{1}{2})}{2^n}$

reverse

$\Rightarrow 2^n\Gamma(n+\frac{1}{2}) = 1, 3, 5 \ldots (2n-1)\sqrt{\pi}$ (Proved)

Q. Evaluate: $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$

Substituting $x^n = \sin^2\theta$

$x = \sin^{2/n}\theta$

$dx = \frac{2}{n}\sin^{(\frac{2}{n}-1)}\theta \cos\theta \, d\theta$

$\therefore I = \int_0^{\pi/2} \frac{\frac{2}{n}\sin^{(\frac{2}{n}-1)}\theta \cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$

$= \frac{2}{n}\int_0^{\pi/2} \sin^{(\frac{2}{n}-1)}\theta \, d\theta$

$= \frac{2}{n}\int_0^{\pi/2} \sin^p\theta \, d\theta$ — ① Let, $p = \frac{2}{n}-1$

but we have,

$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\sqrt{\pi}}{2\Gamma\frac{p+2}{2}}$

from ① we have,

$= \frac{\cancel{2}}{n} \frac{\Gamma\frac{(\frac{2}{n}-1)+1}{2}\sqrt{\pi}}{\cancel{2}\Gamma\frac{(\frac{2}{n}-1)+2}{2}} = \frac{1}{n}\frac{\Gamma\frac{1}{n}\sqrt{\pi}}{\Gamma\frac{\frac{2}{n}+1}{2}}$

$\therefore I = \frac{1}{n} \frac{\Gamma\frac{1}{n}\sqrt{\pi}}{\Gamma\frac{1}{n}+\frac{1}{2}}$

Q. Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

Sol: ~~We have, $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta$~~

~~We know, $\int_0^{\pi/2} \sin^p\theta = \frac{\Gamma\frac{p+1}{2}\sqrt{\pi}}{2\Gamma\frac{p+2}{2}}$~~

$\therefore \int_0^{\pi/2} \sin^{-1/2}\theta = \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\sqrt{\pi}}{2\Gamma\frac{-\frac{1}{2}+2}{2}} = \frac{\Gamma\frac{1}{4}\sqrt{\pi}}{2\Gamma 3/4}$ — ①

We have, $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \, d\theta$

$= \frac{\Gamma\frac{\frac{1}{2}+1}{2}\sqrt{\pi}}{2\Gamma\frac{\frac{1}{2}+1}{2}} = \frac{\Gamma 3/4\sqrt{\pi}}{2\Gamma 1+\frac{1}{4}}$ $\{\Gamma n+1 = n\Gamma n$ $= \frac{\Gamma 3/4 \sqrt{\pi}}{\cancel{2}\frac{1}{\cancel{4}}\Gamma\frac{1}{4}}$

$= \frac{2\Gamma 3/4\sqrt{\pi}}{\Gamma\frac{1}{4}}$ — ⑪

$\therefore \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \frac{\Gamma\frac{1}{4}\sqrt{\pi}}{\cancel{2}\Gamma 3/4} \times \frac{\cancel{2}\Gamma 3/4\sqrt{\pi}}{\Gamma\frac{1}{4}} = (\sqrt{\pi})^2$

$= \pi$

Q. Show that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

L.H.S

$$\int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{\frac12}\theta \cos^{-\frac12}\theta\, d\theta$$

$$= \frac{\Gamma\frac{\frac12+1}{2}\,\Gamma\frac{-\frac12+1}{2}}{2\,\Gamma\frac{\frac12-\frac12+2}{2}} = \frac{\Gamma\frac34\,\Gamma\frac14}{2}$$

$$= \frac{\Gamma\left(1-\frac14\right)\Gamma\frac14}{2}$$

But we have,

$$\Gamma m\, \Gamma(1-m) = \frac{\pi}{\sin m\pi}$$

$\therefore$ LHS

$$\frac{\frac{\pi}{\sin \pi/4}}{2} = \frac{\pi}{2\sin\pi/4} = \frac{\pi}{2\times\frac{1}{\sqrt2}} = \frac{\pi}{\sqrt2}$$

Q. Show that $\Gamma n = \int_0^1 \left(\ln\frac1x\right)^{n-1} dx$; $n>0$

Let, $\ln\frac1x = t$

$\therefore \frac1x = e^t$

$\Rightarrow x = -e^{-t}$

$\Rightarrow dx = -e^{t}dt$

RHS. $= \int_\infty^0 t^{n-1}(-e^{-t}\,dt)$

$= \int_0^\infty t^{n-1} e^{-t}\, dt$

$= \Gamma n$ Proved

Q. Show that $\Gamma n\, \Gamma(1-n) = \int_0^\infty x^{-n}(1+x)^{-1}dx$

Soln:

RHS $= \int_0^\infty \frac{x^{-n}}{(1+x)}\, dx$

Putting $m+n=1$

RHS $= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m,n)$

$\therefore \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \frac{\Gamma(1-n)\,\Gamma n}{\Gamma 1} = \Gamma n\,\Gamma(1-n)$

~~Putting $y = \frac1x$ in the second integral of RHS of eqn (1) we get~~

Q. Show that,

$$\Gamma n = \int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} dx, \; n>0$$

Soln

RHS,

$$\int_0^1 \left(\ln \frac{1}{x}\right)^{n-1} dx$$

Put, $\ln \frac{1}{x} = t$

$\Rightarrow \frac{1}{x} = e^t$

$\Rightarrow x = e^{-t}$

$dx = -e^{-t} dt$.

$$= \int_{\infty}^{0} t^{n-1} \cdot -e^{-t} dt$$

$$= \int_0^{\infty} t^{n-1} e^{-t} dt$$

$$= \Gamma n$$

Q. Show that $\Gamma n \, \Gamma 1-n = \int_0^{\infty} x^{n-1} (1+x)^{-1} dx$.

Soln

RHS

$$= \int_0^{\infty} x^{n-1} (1+x)^{-1} dx$$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx$$

Put, $m+n=1$

$$= \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$

$$= \beta(m,n).$$

But, we know

$$B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \quad \text{\& } m+n=1 \Rightarrow m = 1-n$$

$$= \frac{\Gamma 1-n \, \Gamma n}{\Gamma 1}$$

$$= \frac{\Gamma n \, \Gamma 1-n}{1}$$

$$\Rightarrow B(m,n) = \Gamma n \, \Gamma 1-n$$

⇝ Transcendental equations by Bisection:

A equation containing Polynomial functions, trigonometric function, exponential functions is known as transcendental equation. example, $xe^x = \cos x$.

→ Bisection:

The bisection method of solving a differential equation consists of locating the roots of the equation, $f(x) = 0$, between two numbers x_0 and x_1 such that $f(x)$ is continuous for $x_0 \le x \le x_1$ and also $f(x_0)$ and $f(x_1)$ are of Positive sign.

Let, $f(x_0)$ be negative and $f(x_1)$ Positive. So the approximate of root between them.

$$x = \frac{x_0 + x_1}{2} \quad \text{if } f(x) = 0$$

Then, it asserts that x is the correct root of the given function. On the other hand if $f(x)$ is not equal to zero $\{f(x) \neq 0\}$, then the root lies either between this $\left(x_0, \frac{x_0+x_1}{2}\right)$ or between $\left(\frac{x_0+x_1}{2}, x_1\right)$ depending weather $f(x)$ Positive or negative.

We again bisection the interval and repeat the process until the root is obtain to desire accuracy

Q. find the real root of the $f(x) = x^3 - x - 1 = 0$

Soln

Here, $f(x) = x^3 - x - 1$

Put, $x = 0$

$\Rightarrow f(0) = 0^3 - 0 - 1 = -1 < 0$

Put, $x = 1$

$\Rightarrow f(1) = 1^3 - 1 - 1 = -1 < 0$

Put, $x = 2$

$\Rightarrow f(2) = 2^3 - 2 - 1 = 8 - 3 = 5 > 0$

$\therefore$ Hence the root lies between 1 and 2.

for the first approximation

$$x_0 = \frac{1+2}{2} = \frac{3}{2} = 1.5.$$

$$\therefore f(3/2) = (3/2)^3 - 3/2 - 1$$

$$= \frac{27}{8} - \frac{3}{2} - 1$$

$$= \frac{27-12-8}{8}$$

$$= \frac{7}{8}$$

Hence the roots lies between 1 and 1.5.

$\therefore$ for the 2nd approximation.

$$x_1 = \frac{1+1.5}{2} = \frac{2.5}{2} = \frac{25}{20} = 1.25.$$

$$f(1.25) = (1.25)^3 - 1.25 - 1$$

$$= 1.95 - 1.25 - 1$$

$$= 1.95 - 2.25$$

$$= -0.30$$

2.2500
1.9531

Hence root lies between 1.25 and 1.5

$\therefore$ for 3rd approximation.

$$x_2 = \frac{1.25+1.5}{2}$$

$$= \frac{2.75}{2}$$

$$= 1.3875$$

$$f(1.357) = (1.375)^3 - 1.375 - 1$$

$$= 2.561 - 1.375 - 1$$

$$= 2.561 - 2.375$$

$$= 0.186$$

Hence the root lies between 1.375 and 1.25

$\therefore$ for the 5th approximation.

$$x_4 = \frac{1.375+1.25}{2}$$

$$= 1.34375$$

$$f(1.34375) = (1.34375)^3 - 1.34375 - 1$$

$$= 0.05261$$

$\therefore$ Hence the root lies between 1.34375 and 1.375

for 6th approximation.

$$x_5 = \frac{1.3125+1.34375}{2} = 1.3281$$

$$f(1.3281) = (1.3281)^3 - 1.3281 - 1$$

$$= 0.0144.$$

Hence the root lies between 1.3281 and 1.3125

for 7^{th} approximation.

$$x_6 = \frac{1.3125 + 1.3281}{2}$$

$$= 1.3208.$$

$$f(1.3208) = (1.3208)^3 - 1.3208 - 1$$

$$= 0.028.$$

Hence, the root lies between 1.3125 and 1.3208

the approximate value is = 0

Q. Solve $f(x) = x^3 - 9x + 1 = 0$ using bisect method upto 6^{th} approximation.

Soln

Here, $f(x) = x^3 - 9x + 1 = 0$

Put, $x = 0$

$\Rightarrow f(0) = 0^3 - 9 \times 0 + 1 = 0 + 1 = 1$

Put, $x = 1$

$\Rightarrow f(1) = 1^3 - 9 \times 1 + 1 = 1 - 9 + 1 = -7$

Put, $x = 2$

$\Rightarrow f(2) = 2^3 - 9 \times 2 + 1 = 8 - 18 + 1 = -9$

Put, $x = 3$

$\Rightarrow f(3) = 3^3 - 9 \times 3 + 1 = 27 - 27 + 1 = 1$

So, the root lies between 2 and 3.

for the first approximation.

$$x_0 = \frac{2+3}{2} = \frac{5}{2} = 2.5.$$

~~625~~

$$f(2.5) = (2.5)^3 - 9 \times 2.5 + 1$$

$$= 7.5 - 22.5 + 1$$

$$= -14.$$

Hence, the root lies between 2.5 and 3.

for the 2nd approximation.

$$x_1 = \frac{3+2.5}{2} = 1.75.$$

$$f(1.75) = (1.73)^3 - 9 \times 1.75 + 1$$

$$= 5.3593 - 15.75 + 1$$

$$= -9.3907.$$

Hence the root lies between 1.75 and 2.5.

for 3rd approximation.

$$x_2 = \frac{2.5 + 1.75}{2} = 2.125$$

$$f(2.125) = (2.125)^3 - 9 \times 2.125 + 1$$

$$= 9.5957 - 19.125 + 1$$

$$= -8.5293$$

Hence the root lies between 2.125 and 1.75

for 4th approximation

$$x_3 = \frac{1.75+2.125}{2} = 1.9375.$$

$$f(1.9375) = (1.937)^3 - 9\times1.9375+1$$

$$= 7.2731 - 17.4375 + 1$$

$$= -9.1644.$$

Hence the root lies between 1.9375 and 2.125.

for 5th approximation.

$$x_4 = \frac{2.215+1.9375}{2} = 4.1525.$$

$$f(4.1525) = (4.1525)^3 - 9\times 4.1525 * 1$$

$$= 71.6026 - 37.3725 + 1$$

$$= 35.2301.$$

Hence, the root lies between 4.1525 and 1.9375.

for 6th approximation

$$x_5 = \frac{1.9325+4.1525}{2}$$

$$= 3.045.$$

$$f(3.045) = (3.045)^3 - 9\times 3.045+1$$

$$= 28.2331 - 27.405 + 1$$

$$= 1.8281$$

Hence, the root lies between 3.045 and 4.1525.

for 7th approximation,

$$x_6 = \frac{4.1525+3.045}{2} = 3.59875$$

$$\therefore f(3.59875) = (3.59875)^3 - 9\times3.59875+1$$

$$= 46.607 - 32.388 + 1$$

$$= 15.219$$

⇝ Algebric equations by Bisection :-

This method Provides the approximate solution of the algebric equation of the form,

$$f(x) = 0.$$

Let, the approximate solution is x and exact solution is x_0+h. Where h is very small quantity then we can write,

$$f(x) = 0,$$

$$f(x_0+h) = 0 \quad -①$$

expending this solution by Taylor's series about x, we have.

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{\lfloor 2} f''(x_0) \quad -②$$

this is called Taylor's series.

Since, h is very small quantity.

So, the term containing h^2 and higher Power of h can be neglected, so it should be becomes,

$$f(x_0+h) = f(x_0) + hf'(x_0)$$

$$\Rightarrow 0 = f(x_0) + hf'(x_0).$$

$$\Rightarrow hf'(x_0) = -f(x_0)$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)}$$

this equation gives a Value of h which when added x_0 gives better approximate to the root and let this root be x_1

$$\therefore x_1 = x_0 + h_1 \quad \left\{ \because h_1 = \frac{-f(x_0)}{f'(x_0)} \right\}$$

$$\Rightarrow x_1 = x_0 + \frac{-f(x_0)}{f'(x_0)}$$

Now, using x_1 in case of x_0 and Proceeding exactly in the same way. we have,

$$\therefore x_2 = x_1 + h_2 \quad \left\{ \because h_2 = \frac{-f(x_1)}{f'(x_1)} \right\}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

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