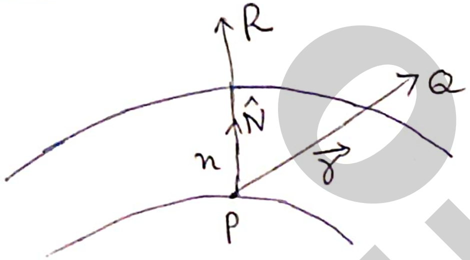
Vector calculus

Vector Differentiation:

Directional derivatives and normal derivatives:

($\vec{N}$) Normal vector:



$\phi(x,y,z)$

$\vec{N} = \nabla\phi$

$\boxed{\vec{N} = \text{vector}}$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

→ The component of a vector $\vec{r}$ along PQ. ~~along~~ PQ.

Directional Derivative $= \nabla\phi \cdot \hat{r}$.

Directional derivatives (DD) = maximum rate of change of ϕ

$\boxed{D.D = \nabla\phi \cdot \hat{r}.}$

$$DD = \frac{\delta\phi}{\delta n}\hat{r}.$$

Q. find the direction derivative of $x^2y^2z^2$ at Point $(1,1,-1)$ in the direction of the tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$

Soln Given, $\phi = x^2y^2z^2$

$$\left[\vec{T} = \text{tangent vector} = \frac{d\vec{r}}{dt}\right]$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ (curve).

$\therefore \vec{r} = e^t\hat{i} + (\sin 2t + 1)\hat{j} + \hat{k}(1 - \cos t)$

$$\frac{d\vec{r}}{dt} = e^t\hat{i} + (2\cos 2t)\hat{j} + (\sin t)\hat{k}.$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=0} = \hat{i} + 2\hat{j} = \vec{T}$$

$$\hat{T} = \frac{\hat{i} + 2\hat{j}}{\sqrt{1^2 + 2^2}} = \frac{\hat{i} + 2\hat{j}}{\sqrt{5}}$$

$$\therefore \nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta z}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2y^2z^2)$$

$= y^2z^2(2x)\hat{i} + x^2z^2(2y)\hat{j} + y^2x^2(2z)\hat{k}$

$= 2xy^2z^2\hat{i} + 2yx^2z^2\hat{j} + 2x^2y^2z\hat{k}.$

$= 2(1)(1)^2(-1)^2\hat{i} + 2(1)(1)^2(-1)^2\hat{j} + 2(1)^2(1)^2(-1)\hat{k}$

$= 2\hat{i} + 2\hat{j} - 2\hat{k}.$

$\therefore$ Directional derivatives $= \nabla\phi \cdot \hat{T}$

$$= (2\hat{i}+2\hat{j}-2\hat{k}) \cdot \frac{\hat{i}+2\hat{j}}{\sqrt{5}}$$

$$= \frac{2+4}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

Q. find the unit normal vector to the surface $x^2+y^2=z$ at a Point $(1,2,5)$.

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2+y^2-z)$$

$$\vec{N} = \nabla\phi = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$(\vec{N})_{(1,2,5)} = 2(1)\hat{i} + 2(2)\hat{j} - \hat{k}$$

$$= 2\hat{i} + 4\hat{j} - \hat{k}$$

$$\therefore \hat{N} = \frac{\vec{N}}{|\vec{N}|} = \frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{4+16+1}} = \frac{2\hat{i}+4\hat{j}-\hat{k}}{\sqrt{21}}$$

Q. Calculate the Directional derivative of $f = xy^2+yz^3$ at the Point $(1,-1,1)$ in the direction of $(3,1,-1)$

$$\vec{d} = 3\hat{i}+\hat{j}-\hat{k} \qquad f = \phi = xy^2+yz^2$$

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{j}\right)(xy^2+yz^3)$$

$$= \frac{\delta}{\delta x}(xy^2+yz^3)\hat{i} + \frac{\delta}{\delta y}(xy^2+yz^3)\hat{j} + \frac{\delta}{\delta z}(xy^2+yz^3)\hat{k}$$

$$\Rightarrow \nabla\phi = y^2\hat{i} + (2xy+z^3)\hat{j} + 3yz^2\hat{k}$$

$$\Rightarrow (\nabla\phi)_{(1,-1,1)} = \hat{i} + (-2+1)\hat{j} - 3\hat{k} = \hat{i} - \hat{j} - 3\hat{k}$$

$$\hat{d} = \frac{3\hat{i}+\hat{j}-\hat{k}}{\sqrt{1^2+(-1)^2+(-3)^2}} = \frac{3\hat{i}+\hat{j}-\hat{k}}{\sqrt{11}}$$

$$\therefore D.D = \nabla\phi \cdot \hat{d} = \hat{i}-\hat{j}-3\hat{k} \cdot \frac{3\hat{i}+\hat{j}-\hat{k}}{\sqrt{11}}$$

$$= \frac{3-1+3}{\sqrt{11}}$$

$$= \frac{6-1}{\sqrt{11}} = \frac{5}{\sqrt{11}}$$

Q. find the directional derivative of the scalar function $f = xyz$ in the direction of outer normal to the surface $z = xy$ at the Point $(3,1,3)$.

$\phi_1 = xyz$ $\qquad D.D = \nabla\phi_1 \cdot d = \nabla\phi_1 \cdot \frac{\vec{d}}{|\vec{d}|} = \nabla\phi_1 \frac{\nabla\phi_2}{|\nabla\phi_2|}$

$\phi_2 = xy - z$

$$\nabla\phi_1 = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(xyz)$$

$$\Rightarrow \nabla\phi_1 = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$= \cancel{y2\hat{i} + xz\hat{j} + xy\hat{k}}$

$$\nabla\phi_2 = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(xy - z)$$

$$= y\hat{i} + x\hat{j} + (-1)\hat{k}.$$

$$\nabla\phi_2 = y\hat{i} + x\hat{j} - \hat{k}.$$

$$\Rightarrow \boxed{\nabla\phi_2 = \hat{i} + 3\hat{j} - \hat{k}}$$

$$\nabla\phi_1 = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$= (1\times 3)\hat{i} + (3\times 3)\hat{j} + (3\times 1)\hat{k}$$

$$\Rightarrow \nabla\phi_1 = 3\hat{i} + 9\hat{j} + 3\hat{k}$$

$$\therefore D.D = \nabla\phi_1 \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$= 3\hat{i} + 9\hat{j} + 3\hat{k} \cdot \frac{\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{1+9+1}}$$

$$= \frac{3 + 27 - 3}{\sqrt{11}}$$

$$= \frac{27}{\sqrt{11}}$$

Q. What is the greatest rate of increase (Directional D.) of the func^n $u = xyz^2$ at the Point $(1, 0, 3)$.

Soln $D.D. = \nabla\phi \cdot \hat{d}$.

$$\nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(xyz^2)$$

$$= yz^2\hat{i} + xz^2\hat{j} + 2xyz\hat{k}$$

$$= 0.\hat{i} + 9\hat{j} + 0\hat{k}$$

$$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{0.\hat{i} + 9\hat{j} + 0.\hat{k}}{\sqrt{9^2}} = \frac{9\hat{j}}{9}$$

$$D.D = 9\hat{j} \cdot \frac{9\hat{j}}{9} = \frac{81}{9} = 9$$

Gradient of scaler field and its geometrical interpretation:-

Gradient of ϕ → scalar field

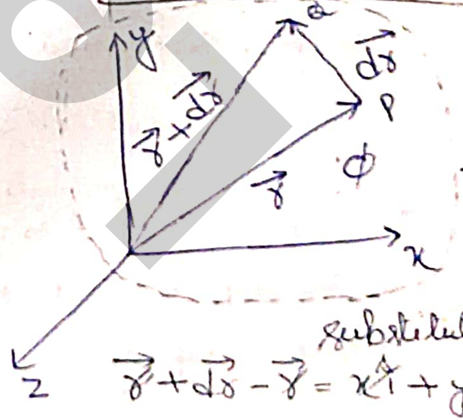
↓

grad ϕ

$$\boxed{\nabla\phi = \text{Vector}}$$

$$\left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi = \text{vector} = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}.$$

$$\boxed{d\phi = \nabla\phi\, dr}$$



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{—(1)}$$

$$\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k} \quad \text{—(2)}$$

$$\vec{r} + \vec{dr} = x\hat{i} + y\hat{j} + z\hat{k} + dx\hat{i} + dy\hat{j} + dz\hat{k} \quad \text{—(3)}$$

substituting eq^n (3) by (1)

$$\vec{r} + \vec{dr} - \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} + dx\hat{i} + dy\hat{j} + dz\hat{k} - x\hat{i} - y\hat{j} - z\hat{k}$$

$\Rightarrow \vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

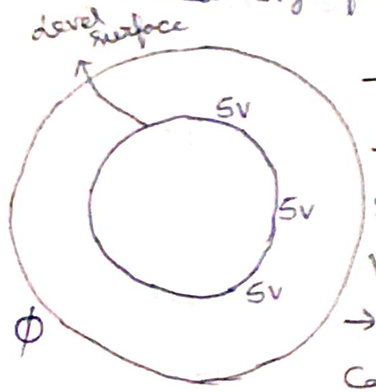
$$d\phi = \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$$

$$\Rightarrow d\phi = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right)(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\Rightarrow d\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\Rightarrow \boxed{d\phi = \underbrace{\nabla\phi}_{\text{grad }\phi}\,\vec{dr}}$$

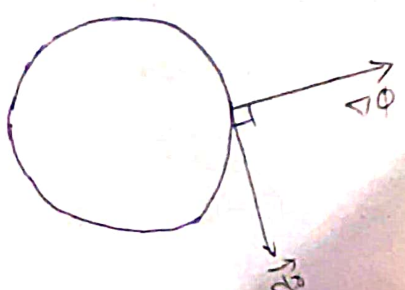
Geometrical significance of Grad ϕ :-



→ Let us consider a surface ϕ

→ Join all such points in the surface which have same values.

→ And the surface so formed is called level surface.

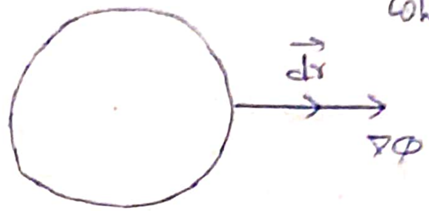


→ The gradient of ϕ is normal to the displacement dr & also the surface

$\therefore d\phi = \nabla\phi\, dr$

$= \nabla\phi\, dr \cos\theta$

$\Rightarrow d\phi = \nabla\phi\, dr \cos 90^\circ = 0$



When $\vec{dr}$ is along $\nabla\phi$ then –

$d\phi = \nabla\phi\, dr$

$= \nabla\phi\, dr \cos\theta$

$= \nabla\phi\, dr \cos 0^\circ$

$$\Rightarrow \left(\frac{d\phi}{dr}\right)_{max} = \nabla\phi$$

→ The gradient of scalar field (ϕ) is a vector. Whose magnitude is maximum & is equal to rate of change of ϕ w.r.to $\vec{dr}$.

Q. Calculate grad of ϕ if $\phi = xyz$

Soln Grad $\phi = \nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)xyz$

$= yz\hat{i} + xz\hat{j} + xy\hat{k}$

Q. Calculate grad of ϕ if $\phi = \log(x^2+y^2+z^2)$

Soln Grad $\phi = \nabla\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\{\log(x^2+y^2+z^2)\}$

$$= \frac{2x}{x^2+y^2+z^2}\hat{i} + \frac{2y}{x^2+y^2+z^2}\hat{j} + \frac{2z}{x^2+y^2+z^2}\hat{k}$$

$$= \frac{2}{(x^2+y^2+z^2)}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$\text{grad}\,\phi = \frac{2}{x^2+y^2+z^2}\vec{r}$$

$$= \frac{2}{r^2}\vec{r}$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|r^2| = \sqrt{x^2+y^2+z^2}$

Divergence of a vector function :

The divergence of vector Point f^n $\vec{F}$ is denoted by div F and defined as -

$$F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

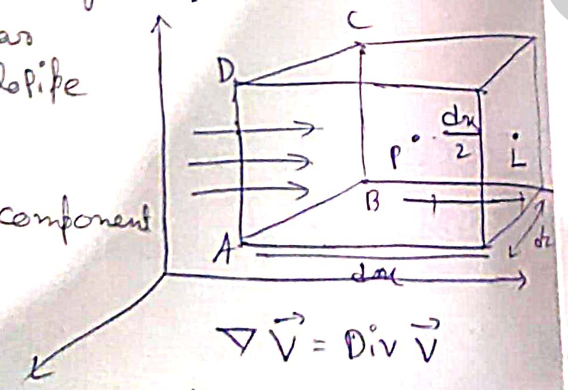
$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

∴ div F is Scalar f^n.

Physical interpretation of divergence :

Let us consider a rectangular Parallelopipe ABCDEFGH.



Vel. of fluid along x-component

$$= V_x + \frac{\delta V_x}{\delta x}\frac{dx}{2}$$

= Velocity + Rate of change of velocity × Dist. b/w Points P and L

$$\nabla\vec{V} = \text{Div}\,\vec{V}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

[flux = Velocity × area.]

Out going flux EHFG = Velocity of x-com × area

{Area EFGH = l × b = dy×dz}

$$= V_x + \frac{\delta V_x}{\delta x}\frac{dx}{2} \times dz\,dy \text{ — (1)}$$

Incoming flux (ABCD) = Velocity of x-com × area

Area of (ABCD) = l×b = dy×dz

$$= V_x - \frac{\delta V_x}{\delta x}\frac{dx}{2} \times dy\,dz \text{ — (11)}$$

Net flux of x - ~~direction~~ = (1) − (11) ⇒

$$= \cancel{V_x} + \frac{\delta V_x}{\delta x}\cdot\frac{dx}{2} \times dy\,dz - \cancel{V_x} + \frac{\delta V_x}{\delta x}\cdot\frac{dx}{dz} \times dy\,dz$$

$$= \frac{\cancel{2}\,\delta V_x\,dx\,dy\,dz}{\cancel{2}\,\delta x}$$

$$= \frac{\delta V_x}{\delta x}dx\,dy\,dz \text{ — (3)}$$

Net flux of y-direction $= \frac{\delta V_y}{V_y}dx\,dy\,dz$ — (4)

Net flux of z direction $= \frac{\delta V_z}{V_z}dx\,dy\,dz$ — (5)

Net outward flux ABCDEFGH, 3 + 4 + 5

$$= \frac{\delta V_x}{\delta x} V_x V_y V_z + \frac{\delta V_y}{\delta y} V_x V_y V_z + \frac{\delta V_z}{\delta z} V_x V_y V_z$$

$$= V_x V_y V_z \left(\frac{\delta V_x}{\delta x} + \frac{\delta V_y}{\delta y} + \frac{\delta V_z}{\delta z} \right)$$

outward flux per unit volume of parallelopiped.

$$\frac{\text{flux}}{\text{Volume}} = \frac{V_x V_y V_z \left(\frac{\delta V_x}{\delta x} + \frac{\delta V_y}{\delta y} + \frac{\delta V_z}{\delta z} \right)}{V_x V_y V_z}$$

$$= \left(\frac{\delta V_x}{\delta x} + \frac{\delta V_y}{\delta y} + \frac{\delta V_z}{\delta z} \right)$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k} \right) \left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \right)$$

$$= \nabla . \vec{V} = \text{Div}. \vec{V}$$

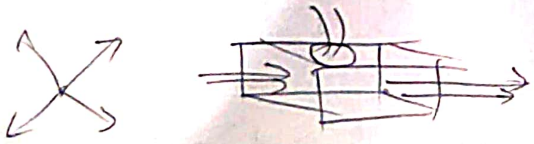
Cases

(I) $\text{Div} \vec{A} = \nabla \vec{A}$

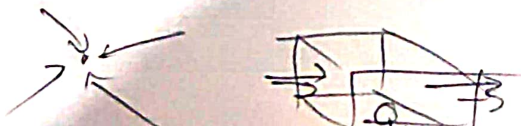
$\therefore \nabla . \vec{A} = 0$ $\quad \boxed{\vec{A} = \text{Solenoid}}$

(II) $\text{Div} \vec{A} > 0$

$\text{Div} \vec{A} = +\text{ve}$



(III) $\text{Div} \vec{A} = -\text{ve}$



If the fluid is compressible, there can be no gain or loss in the volume element. Hence,

$$\text{div } \vec{V} = 0 \quad \text{—(1)}$$

and v is called a solenoidal vector function.

Eqn (1) is also called the equation of continuity or conservation of mass.

Q. If $u = x^2 + y^2 + z^2$ & $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\text{Div}(u\vec{r})$ in terms of u.

Soln

$$\text{Div}(u\vec{r}) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k} \right) . (x^2 + y^2 + z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k} \right) \{ (x^3 + y^2x + z^2x)\hat{i} + (x^2y + y^3 + zy^2)\hat{j} + (x^2z + zy^2 + z^2)\hat{k}$$

$$= \frac{\delta}{\delta x}(x^3 + xy^2 + z^2x) + \frac{\delta}{\delta y}(x^2y + y^3 + zy^2) + \frac{\delta}{\delta z}(x^2z + zy^2 + z^3)$$

$$= 3x^2 + y^2 + z^2 + x^2 + 3y^2 + 2zy + x^2 + y^2 + 2z^2$$

$$= 5x^2 + 5y^2 + 5z^2$$

$$= 5(x^2 + y^2 + z^2) = 5u.$$

Q. Check that vector $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$. is solenoidal or not?

Soln

$$\text{Div } \vec{V} = \vec{\nabla}.\vec{V} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right).\{(x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}\}$$

$$= \frac{\delta}{\delta x}(x+3y) + \frac{\delta}{\delta y}(y-3z) + \frac{\delta}{\delta z}(x-2z)$$

$$= 1 + 1 - 2$$

$$= 0$$

$\therefore$ The given vector is ~~divergent~~ solenoidal.

Q. find the value of n for which the vector $r^n\vec{r}$ is solenoidal.

Soln

$r^n\vec{r}$ = Solenoidal

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r^2 = x^2+y^2+z^2$$
$$\Rightarrow r = \sqrt{x^2+y^2+z^2}$$
$$\Rightarrow r^n = (x^2+y^2+z^2)^{n/2}$$

$\Rightarrow \nabla r^n\vec{r} = 0$

$\Rightarrow \text{Div } r^n\vec{r} = 0$

$$\Rightarrow \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2+y^2+z^2)^{n/2}\,(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right).(x^2+y^2+z^2)^{n/2}.x\hat{i} + (x^2+y^2+z^2)^{n/2}y\hat{j} + (x^2+y^2+z^2)^{n/2}z\hat{k} = 0$$

$$\Rightarrow \frac{\delta}{\delta x}(x^2+y^2+z^2)^{n/2}x + \frac{\delta}{\delta y}(x^2+y^2+z^2)^{n/2}y + \frac{\delta}{\delta z}(x^2+y^2+z^2)^{n/2}z = 0$$

$$\Rightarrow \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2x(x) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2y(y) + (x^2+y^2+z^2)^{\frac{n}{2}} + \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2z(z) + (x^2+y^2+z^2)^{\frac{n}{2}} = 0$$

$$\Rightarrow 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}x^2 + n(x^2+y^2+z^2)^{\frac{n}{2}-1}y^2 + n(x^2+y^2+z^2)^{\frac{n}{2}-1}z^2 = 0$$

$$\Rightarrow 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x^2+y^2+z^2)' = 0$$

$$\Rightarrow 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}}\cancel{(x^2+y^2+z^2)^{-1}(x^2+y^2+z^2)} = 0$$

$$\Rightarrow 3(x^2+y^2+z^2)^{\frac{n}{2}} + n(x^2+y^2+z^2)^{\frac{n}{2}} = 0$$

$$\Rightarrow \boxed{3+n}(x^2+y^2+z^2)^{\frac{n}{2}} = 0$$

$$\Rightarrow n = -3$$

Q. $\text{Div}\left(\frac{\vec{r}}{r^2}\right) = ?$

Soln

$$\nabla.\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}\right)$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left(\frac{x}{x^2+y^2+z^2}\hat{i} + \frac{y}{x^2+y^2+z^2}\hat{j} + \frac{z}{x^2+y^2+z^2}\hat{k}\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{x}{x^2+y^2+z^2}\right) + \frac{\delta}{\delta x}\left(\frac{y}{x^2+y^2+z^2}\right) + \frac{\delta}{\delta x}\left(\frac{z}{x^2+y^2+z^2}\right)$$

$$\Rightarrow \frac{(x^2+y^2+z^2)\frac{\delta x}{\delta x} - x\frac{\delta}{\delta x}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)\frac{\delta y}{\delta y} - y\frac{\delta}{\delta y}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)\frac{\delta z}{\delta z} - z\frac{\delta}{\delta x}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2 - x(2x)}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - y(2y)}{(x^2+y^2+z^2)^2} + \frac{x^2+y^2+z^2 - z(2z)}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2 - 2x^2 + x^2+y^2+z^2 - 2y^2 + x^2+y^2+z^2 - 2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

$$= \frac{1}{r^2}$$

Q. Show that $\nabla(\phi A) = \nabla\phi . A + \phi(\nabla . A)$

L.H.S

$$\nabla.(\phi A) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left(A_x\phi\,\hat{i} + A_y\phi\,\hat{j} + A_z\phi\,\hat{k}\right)$$

$$= \frac{\delta}{\delta x}(A_x\phi) + \frac{\delta}{\delta y}(A_y\phi) + \frac{\delta}{\delta z}(A_z\phi)$$

$$= A_x\frac{\delta\phi}{\delta x} + \phi\frac{\delta A_x}{\delta x} + A_y\frac{\delta\phi}{\delta y} + \phi\frac{\delta A_y}{\delta y} + \phi\frac{\delta}{\delta z}A_z + A_z\frac{\delta}{\delta z}\phi$$

$$\Rightarrow \nabla.(\phi A) = \phi\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}\right) + \left(A_x\frac{\delta\phi}{\delta x} + A_y\frac{\delta\phi}{\delta y} + A_z\frac{\delta\phi}{\delta z}\right)$$

R.H.S

$$\nabla\phi.A + \phi(\nabla.A) = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right).\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) + \phi\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta}{\delta z}(A_z)\right)$$

$$= \left(A_x\frac{\delta\phi}{\delta x} + A_y\frac{\delta\phi}{\delta y} + A_z\frac{\delta\phi}{\delta z}\right) + \phi\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta y}\right)$$

$$= \phi\left(\frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z}\right) + \left(A_x\frac{\delta\phi}{\delta x} + A_y\frac{\delta\phi}{\delta y} + A_z\frac{\delta\phi}{\delta z}\right)$$

$\therefore$ L.H.S = R.H.S

→ Curl ÷

The curl of the vector Point function F is defined as below

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta F_3}{\delta y} - \frac{\delta F_2}{\delta z}\right) - \hat{j}\left(\frac{\delta F_3}{\delta x} - \frac{\delta F_1}{\delta z}\right) + \hat{k}\left(\frac{\delta f_2}{\delta x} - \frac{\delta F_1}{\delta y}\right).$$

→ curl $\vec{F}$ is a vector quantity.

→ Physical meaning of curl ÷

We know that $\vec{V} = \vec{\omega} \times \vec{r}$, where ω is angular velocity, $\vec{V}$ is the the linear velocity and $\vec{r}$ is the Position vector of a Point on the rotating body.

$$\text{curl } \vec{V} = \vec{\nabla} \times \vec{V}$$

$$\text{curl } \vec{V} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

$$= \vec{\nabla} \times \left[(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})\right]$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta y}\hat{k}\right) \times \left[(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$\cancel{= \hat{i}\left\{\frac{\delta}{\delta y}(\omega_1 y - \omega_2 x)\right\} - \hat{j}\left\{\frac{\delta}{\delta x}(\omega_1 y - \omega_2 x) - \frac{\delta}{\delta z}(\omega_3 x - \omega_1 z)\right\} \frac{\delta}{\delta z}(\omega_2 z - \omega_3 y)\}}$$

$$\cancel{+ \hat{k}\left(\frac{\delta}{\delta x}(\omega_3 x - \omega_1 z) - \frac{\delta}{\delta y}(\omega_2 z - \omega_3 y)\right)}$$

$$\cancel{= \omega_1\hat{i} - \omega_2\frac{\delta y}{\delta y}\hat{i} - \frac{\omega_3 \delta x}{\delta z}\hat{i} + \omega_2\hat{i}}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(\omega_1 y - \omega_2 x) - \frac{\delta}{\delta z}(\omega_3 x - \omega_1 z)\right] - \hat{j}\left[\frac{\delta}{\delta x}(\omega_1 y - \omega_2 x) - \frac{\delta}{\delta z}(\omega_2 z - \omega_3 y)\right] + \hat{k}\left[\frac{\delta}{\delta x}(\omega_3 x - \omega_1 z) - \frac{\delta}{\delta y}(\omega_2 z - \omega_3 y)\right]$$

$= \hat{i}\left[\omega_1 - \ldots + \omega_1\right] - \hat{j}\left[\ldots - \omega_2 - \omega_2\right] + \hat{k}(\omega_3 + \omega_3)$

$= \hat{i}(\omega_1 + \omega_1) - \hat{j}(-\omega_2 - \omega_2) + \hat{k}(\omega_3 + \omega_3)$

$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) = 2\vec{\omega}$

If curl $\vec{F} = 0$, the field F is termed as irrotational.

Mathematical expression of curl:

$$\text{curl } \vec{A} = \frac{\oint \vec{A} \cdot \vec{dl}}{\text{Area } ABCD} = \vec{\nabla} \times \vec{A}$$

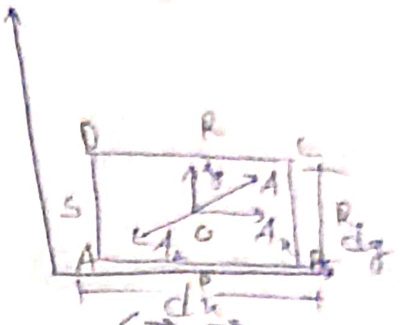
Note:

(i) $\nabla \phi$ = Scalar

(ii) $\nabla \cdot \vec{A}$ = Vector

(iii) $\vec{\nabla} \times \vec{A}$ = Vector.

Physical interpretation of curl:



$$\text{curl } \vec{A} = \frac{\oint \vec{A} \cdot \vec{dl}}{\text{Area } ABCD} \quad \text{— (i)}$$

$(\text{Along } AB)_P$ = Component of $\vec{A}$ along BA − Change (distance $O \to P$)

$$= A_x - \frac{\delta A_x}{\delta y} \times \left(\frac{dy}{2}\right) \quad \text{— (ii)}$$

$(\text{Along } DC)_R$ = Component of $\vec{A}$ along CD + change (Dist $O \to R$)

$$= A_x + \frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right) \quad \text{— (iii)}$$

$$(\text{Along } BC)_Q = A_y + \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right) \quad \text{— (iv)}$$

$$(\text{Along } DA)_S = A_y + \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right) \quad \text{— (v)}$$

$$\text{Area of } ABCD = dx \times dy \quad \text{— (vi)}$$

$\therefore \oint \vec{A} \cdot \vec{dl} = (\text{Along } AB)\, AB + (\text{Along } BC)\, BC - (\text{Along } CD)\, CD - (\text{Along } DA)\, DA$

$$= \left[A_x + \frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)\right] dx + \left[A_x + \frac{\delta A_y}{\delta y}\left(\frac{dx}{2}\right)\right] dy$$

$$-\left[\frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)\right]dx - \left[A_y + \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right)\right]dy$$

$$= \cancel{dx A_x} - \frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)dx + A_y dy + \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right)dy -$$

$$\cancel{A_x dx} - \frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)dx - A_y dy + \left(\frac{\delta A_y}{\delta x}\right)\left(\frac{dx}{2}\right)dy$$

$$= -\frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)dx - \frac{\delta A_x}{\delta y}\left(\frac{dy}{2}\right)dx + \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right)dy$$

$$+ \frac{\delta A_y}{\delta x}\left(\frac{dx}{2}\right)dy$$

$$= -\frac{\cancel{2}\,\delta A_x\, dy\, dx}{\cancel{2}\,\delta y} + \frac{\cancel{2}\,\delta A_y\, dx\, dy}{\cancel{2}\,\delta x}$$

$$= dx\,dy\left[-\frac{\delta A_x}{\delta y} + \frac{\delta A_y}{\delta x}\right]\hat{k}$$

$$= dx\,dy\left[\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right]\hat{k}.$$

$$\therefore \oint_{ABCD} \vec{A}\, d\vec{l} = \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)dx\,dy\,\hat{k}.$$

$$\text{curl}\,\vec{A} = \frac{\oint_{ABCD} \vec{A}\cdot d\vec{l}}{\text{Area of } ABCD} = \frac{\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)dx\,dy}{dx\,dy}$$

$$\therefore \text{curl}\,\vec{A} = \left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)\hat{k}$$

Q. If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ then find curl $\vec{A}$.

Soln

$$\vec{\nabla}\times\vec{A} = \text{curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta A_x}{\delta y} - \frac{\delta A_y}{\delta z}\right) - \hat{j}\left(\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta x}\right) + \hat{k}\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right)$$

Q. If $\vec{\nabla}\times\vec{A} = 0$ then $\vec{A}$ is

(a) Irrolational (b) Solenoidal (c) Rotational (d) None.

↳ No Rotational.

Q. If $\vec{V} = xy^2\hat{i} + 2yx^2z\hat{j} - 3yz^2\hat{k}$ then curl $\vec{V}$ at Point $(1,-1,1)$:-

i) $-(\hat{j}+2\hat{k})$ ii) $(\hat{i}+3\hat{k})$ iii) $-(\hat{i}+2\hat{k})$ iv) 0.

Soln

$$\text{curl}\,\vec{V} = \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy^2 & 2yx^2z & -3yz^2 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta}{\delta y}(-3yx^2) - \frac{\delta}{\delta z}(2yx^2z)\right) -$$

$$\hat{j}\left(\frac{\delta}{\delta x}(-3y^2) - \frac{\delta}{\delta z}(xy^3)\right) + \hat{k}\left(\frac{\delta}{\delta x}(2yx^2z) - \frac{\delta}{\delta y}\right.$$

$$= \hat{i}(-3z^2 - 2yx^2) - 0\hat{j} + (4xyz - 2xy)\hat{k}$$

$$= \hat{i}(-3(1)^2 - 2(-1)(1)^2) - 0\hat{j} + (4(1)(-1) - 2(1)(1))\hat{k}$$

$$= -\hat{i} - 2\hat{k}$$

Ans iii. $(-\hat{i} + 2\hat{k})$

Laplacian operator (∇^2)

The scalar differential operator $\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$ is known as Laplacian operator.

$$\therefore \nabla^2 = \nabla.\nabla = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right).\left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)$$

$$= \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$$

Note :-

① $\nabla^2\phi = \text{div}(\text{grad}\,\phi) = \nabla.\nabla\phi$

$$= \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\phi$$

(ii) $\nabla^2\phi = 0$ Laplacian equation.

Vector identities :-

① div. grad $\phi = \nabla^2\phi$

Proof :-

L.H.S

div. grad ϕ

$$= \vec{\nabla}.(\vec{\nabla}\phi)$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi$$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right)$$

$$= \frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} + \frac{\delta^2\phi}{\delta z^2}$$

$$= \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)\phi$$

$$= \nabla^2\phi = \text{R.H.S (Proved)}$$

② Prove that curl grad $\phi = 0$

Proof :-

grad $\phi = \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi$

$$\Rightarrow \text{grad}\,\phi = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta y}\hat{k}\right).$$

Now, L.H.S curl grad $\phi = \vec{\nabla}\times(\vec{\nabla}\phi)$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta x}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta \phi}{\delta x} & \frac{\delta \phi}{\delta y} & \frac{\delta \phi}{\delta z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}\left(\frac{\delta \phi}{\delta z}\right) - \frac{\delta}{\delta z}\left(\frac{\delta \phi}{\delta y}\right)\right] - \hat{j}\left[\frac{\delta}{\delta x}\left(\frac{\delta \phi}{\delta z}\right) - \frac{\delta}{\delta z}\left(\frac{\delta \phi}{\delta x}\right)\right]$$

$$+ \hat{k}\left[\frac{\delta}{\delta x}\left(\frac{\delta \phi}{\delta y}\right) - \frac{\delta}{\delta y}\left(\frac{\delta \phi}{\delta x}\right)\right]$$

$$= \hat{i}\left[\frac{\delta^2 \phi}{\delta y \delta z} - \frac{\delta^2 \phi}{\delta y \delta z}\right] - \hat{j}\left[\frac{\delta^2 \phi}{\delta x \delta z} - \frac{\delta^2 \phi}{\delta x \delta z}\right] + \hat{k}\left[\frac{\delta^2 \phi}{\delta x \delta y} - \frac{\delta^2 \phi}{\delta x \delta y}\right]$$

$$= \hat{i}.0 - \hat{j}.0 + \hat{k}.0$$

$= 0$ R.H.S Proved.

③ Prove that div curl $\vec{F} = 0$

Proof ÷

Let, $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$

curl $\vec{F} = \vec{\nabla} \times \vec{F}$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)\left(F_x \hat{i} + F_y \hat{j} + F_z \hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \hat{j}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

Now, L.H.S

div curl $\vec{F}$

$= \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left[\hat{i}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \hat{j}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \hat{k}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)\right]$$

$$= \frac{\delta}{\delta x}\left(\frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z}\right) - \frac{\delta}{\delta y}\left(\frac{\delta F_z}{\delta x} - \frac{\delta F_x}{\delta z}\right) + \frac{\delta}{\delta z}\left(\frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}\right)$$

$$= \frac{\delta^2 F_z}{\delta x \delta y} - \frac{\delta^2 F_y}{\delta x \delta z} - \frac{\delta^2 F_z}{\delta x \delta y} + \frac{\delta^2 F_x}{\delta y \delta z} + \frac{\delta^2 F_y}{\delta x \delta z} - \frac{\delta^2 F_x}{\delta y \delta z}$$

$= 0$ R.H.S (Proved)

④ Prove that $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B}(\vec{\nabla} \times \vec{A}) - \vec{A}(\vec{\nabla} \times \vec{B})$

Proof ÷ L.H.S $\vec{\nabla} \cdot (A \times \vec{B})$

$= \Sigma \hat{i} \frac{\delta}{\delta x}(\vec{A} \times \vec{B})$

$$\boxed{\vec{\nabla} = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k} = \Sigma \hat{i} \frac{\delta}{\delta x}}$$

$$= \sum \hat{i} \cdot \left(\frac{\delta \vec{A}}{\delta x} \times \vec{B} + \vec{A} \times \frac{\delta \vec{B}}{\delta x} \right)$$

$$= \sum \hat{i} \cdot \frac{\delta \vec{A}}{\delta x} \times \vec{B} + \sum \hat{i} \cdot \vec{A} \times \frac{\delta \vec{B}}{\delta x}$$

$$= \sum \hat{i} \cdot \frac{\delta \vec{A}}{\delta x} \vec{B} - \sum \hat{i} \frac{\delta \vec{B}}{\delta x} \times \vec{A}$$

$$= \left(\sum \hat{i} \times \frac{\delta \vec{A}}{\delta x} \right) \cdot \vec{B} - \left(\sum \hat{i} \times \frac{\delta \vec{B}}{\delta x} \right) \cdot \vec{A}$$

$$\boxed{\begin{aligned} &\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} \\ &\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} \\ &= \left(\sum \hat{i} \frac{\delta}{\delta x} \right) \times \vec{A} \\ &= \sum \hat{i} \frac{\delta \vec{A}}{\delta x} \end{aligned}}$$

$$= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - (\vec{\nabla} \times \vec{B}) \cdot \vec{A}$$

$$= \vec{B} (\vec{\nabla} \times \vec{A}) - \vec{A} (\vec{\nabla} \times \vec{B})$$

R.H.S (Proved)

⑤ Prove that $\text{div}(\phi \vec{V}) = \phi (\text{div } \vec{V}) + (\text{grad } \phi) \cdot \vec{V}$.

Proof:

Let, $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$

$\text{div}(\phi \vec{V}) = \vec{\nabla} (\phi \vec{V})$

$$= \left(\frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k} \right) \left[\phi (V_x \hat{i} + V_y \hat{j} + V_z \hat{k}) \right]$$

$$= \frac{\delta (\phi V_x)}{\delta x} + \frac{\delta (\phi V_y)}{\delta y} + \frac{\delta (\phi V_z)}{\delta z}$$

$$= \phi \frac{\delta V_x}{\delta x} + V_x \frac{\delta \phi}{\delta x} + \phi \frac{\delta V_y}{\delta y} + V_y \frac{\delta \phi}{\delta x} + \phi \frac{\delta V_z}{\delta z} + V_z \frac{\delta \phi}{\delta z}$$

$$= \phi \left(\frac{\delta V_x}{\delta x} + \frac{\delta V_y}{\delta y} + \frac{\delta V_z}{\delta z} \right) + \left(\frac{\delta \phi}{\delta x} V_x + \frac{\delta \phi}{\delta y} V_y + \frac{\delta \phi}{\delta z} V_z \right)$$

$$= \phi \left[\left(\hat{i} \frac{\delta}{\delta x} + \hat{j} \frac{\delta}{\delta y} + \hat{k} \frac{\delta}{\delta z} \right) \cdot (V_x \hat{i} + V_y \hat{j} + V_z \hat{k}) \right] + \left[\left(\hat{i} \frac{\delta \phi}{\delta x} + \hat{j} \frac{\delta \phi}{\delta y} + \hat{k} \frac{\delta \phi}{\delta z} \right) (V_x \hat{i} + V_y \hat{j} + V_z \hat{k}) \right]$$

$$= \phi (\text{div } \vec{V}) + (\text{grad } \phi) \vec{V}$$

Hence (Proved)

⑥ $\text{div}(F \nabla g) = f \nabla^2 g + \nabla F \nabla g$

Step 1:

$$\Delta g = \frac{\delta g}{\delta x} \hat{i} + \frac{\delta g}{\delta y} \hat{j} + \frac{\delta g}{\delta z} \hat{k}$$

Step 2:

$$F \nabla g = F \frac{\delta g}{\delta x} \hat{i} + f \frac{\delta g}{\delta y} \hat{j} + f \frac{\delta g}{\delta z} \hat{k}$$

Step 3: $\text{div}(f \nabla g) = \nabla (f \nabla g)$.

$$\text{div}(f \nabla g) = \nabla \cdot f \nabla g$$

$$= \left(\frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k} \right) \left(F \frac{\delta g}{\delta x} \hat{i} + F \frac{\delta g}{\delta y} \hat{j} + F \frac{\delta g}{\delta z} \hat{k} \right)$$

$$= \frac{\delta}{\delta x} \left(F \frac{\delta g}{\delta x} \right) + \frac{\delta}{\delta y} \left(F \frac{\delta g}{\delta y} \right) + \frac{\delta}{\delta z} \left(F \frac{\delta g}{\delta z} \right)$$

$$= \frac{\delta F}{\delta x} \frac{\delta g}{\delta x} + F \frac{\delta^2 g}{\delta x^2} + \frac{\delta F}{\delta y} \cdot \frac{\delta g}{\delta y} + F \frac{\delta^2 g}{\delta y^2} + \frac{\delta f}{\delta z} \frac{\delta g}{\delta z} + F \frac{\delta^2 g}{\delta z^2}$$

$$= \left(f\frac{\delta^2 g}{\delta x^2} + f\frac{\delta^2 g}{\delta y^2} + f\frac{\delta^2 g}{\delta z^2}\right) + \left(\frac{\delta f}{\delta x}\cdot\frac{\delta g}{\delta x} + \frac{\delta f}{\delta y}\cdot\frac{\delta g}{\delta y} + \frac{\delta f}{\delta z}\cdot\frac{\delta g}{\delta z}\right)$$

$$= f\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2 g}{\delta y^2} + \frac{\delta^2 g}{\delta z^2}\right) g + \left(\hat{i}\frac{\delta f}{\delta x} + \hat{j}\frac{\delta f}{\delta y} + \hat{k}\frac{\delta f}{\delta z}\right)\cdot\left(\hat{i}\frac{\delta g}{\delta x} + \hat{j}\frac{\delta g}{\delta y} + \hat{k}\frac{\delta g}{\delta z}\right)$$

$$= f\nabla^2 g + \nabla f \cdot \nabla g$$

(Proved)

7) $\text{div}(f\nabla g) - \text{div}(g\nabla f) = f\nabla^2 g - g\nabla^2 f.$

Since,

$$\text{div}(f\nabla g) = f\nabla^2 g + \nabla f\,\nabla g.$$

$$\text{div}(g\nabla f) = g\nabla^2 f + \nabla g\,\nabla f.$$

$$\therefore \text{div}(f\nabla g) - \text{div}(g\nabla f).$$

$$= f\nabla^2 g + \nabla f\,\nabla g - [g\nabla^2 f + \nabla g\,\nabla f]$$

$$= f\nabla^2 g + \cancel{\nabla f\,\nabla g} - g\nabla^2 f - \cancel{\nabla g\,\nabla f}$$

$$= f\nabla^2 g - g\nabla^2 f.$$

Vector integration

Ordinary integral of vector :-

$$\int \vec{F}(u)\,du = \left(\int F_1(u)\hat{i} + \int f_2(u)\hat{j} + \int F_3(u)\hat{k}\right)du$$

$\vec{F}$ = force

→ u is scalar quantity.

$$\int \vec{F}(u)\,du = \vec{G} + \vec{C}$$

→ ~~Ident~~ Indefinite Integration [Int. without limits]

Q. Find $\int \vec{F}\cdot\vec{G}\,dt$ where $\vec{F} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ & $\vec{G} = 2t\hat{i} + 6t\hat{k}$. [on the limits 0 to 2]

Soln

$$\int_0^2 \vec{F}\cdot\vec{G}\,dt = \int_0^2 \left(t\hat{i} - t^2\hat{j} + (t-1)\hat{k}\right)\cdot\left(2t\hat{i} + 6t\hat{k}\right)dt$$

$$= \int_0^2 \{2t^2 - 0 + 6t(t-1)\}\,dt$$

$$= \int_0^2 (2t^2 + 6t^2 - 6t)\,dt$$

$$= \int_0^2 (8t^2 - 6t)\,dt$$

$$= \frac{8}{3}\left[t^3\right]_0^2 - \frac{\cancel{6}^{\,3}}{\cancel{2}}\left[t^2\right]_0^2$$

$$= \frac{8\times 8}{3} - 12 = \frac{64-36}{3} = \frac{28}{3}$$

Q. An accelaration of a Particle is $\vec{a} = 8\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j} + 8t\,\hat{k}$. Find the velocity & Displacement at time $t=0$ second.

Sol: Velocity,

$$\vec{V} = \int \vec{a}\,dt = \int (8\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j} + 8t\,\hat{k})\,dt$$

$$\vec{V}(t) = \left[\frac{\overset{4}{\cancel{8}}\sin 2t}{\cancel{2}}\hat{i} + \frac{\overset{2}{\cancel{4}}\cos 2t}{\cancel{2}}\hat{j} + \frac{\overset{4}{\cancel{8}}t^2}{\cancel{2}}\hat{k} + \vec{c}\right]$$

$$\vec{V}(t) = 4\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + 4t^2\hat{k} + \vec{c} \quad — ①$$

$$\vec{V}(0) = 4\sin 0\,\hat{i} + 2\cos 0\,\hat{j} + 4\,0^2\hat{k} + \vec{c}$$

$$\Rightarrow 0 = 0 + 2\hat{j} + 0 + \vec{c}$$

$$\Rightarrow \vec{c} = 2\hat{j} \quad — ②$$

Putting ② in ① we get,

$$\vec{V}(t) = 4\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + 4t^2\hat{k} - 2\hat{j}$$

$$\Rightarrow \vec{V}(t) = 4\sin 2t\,\hat{i} + 2(\cos 2t - 2)\hat{j} + 4t^2\hat{k}$$

Displacement

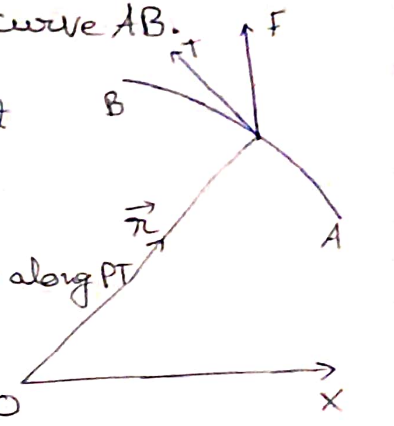
$$\vec{S}(t) = \int \vec{V}(t)\,dt = \int [4\sin 2t\,\hat{i} + 2(\cos 2t - 1)\hat{j} + 4t^2\hat{k}]$$

$$= -4\frac{\cos 2t}{2}\hat{i} + 2\left(\frac{\sin 2t}{2} - t\right)\hat{j} + \frac{4t^3}{3}\hat{k}$$

Line integral :-

Let $\vec{F}(x,y,z)$ be a vector f^n and a curve AB

Line integral of a vector function $\vec{F}$ along the curve AB is defined as integral of the component of $\vec{F}$ along the tangent to the curve AB.



component of $\vec{F}$ along a tangent PT at P

= Dot Product of $\vec{F}$ and unit Vector along PT

$$= \vec{F}\cdot\frac{d\vec{r}}{ds}$$

$$\text{Line integral} = \sum \vec{F}\cdot\frac{d\vec{r}}{ds} \text{ from A to B along the curve}$$

$$\therefore \text{Line integral} = \int_C \left(\vec{F}\cdot\frac{d\vec{r}}{ds}\right)ds = \int_C \vec{F}\cdot d\vec{r}$$

Note:-

$$\boxed{d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}}$$

① If F represents the variable force acting on a Particle arc AB, then total work done.

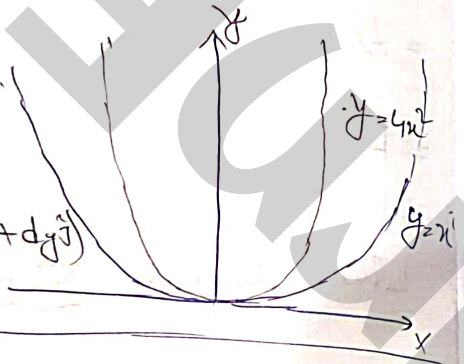
$$W = \int_A^B \vec{F}\cdot d\vec{r}$$

② If $\vec{v}$ represents the velocity of a liquid then

$\oint \vec{V}\cdot d\vec{r}$ is called the <u>circulation</u> of V round the closed curve C.

③ When the Path of integration is a closed the notation of integration is $\oint$ in Place of $\int$.

Q. If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a Particle in x-y Plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done.



<u>Sol</u>

$$W = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C (2x^2y\,\hat{i} + 3xy\,\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C 2x^2y\,dx + 3xy\,dy$$

$$y = 4x^2$$

$$\frac{dy}{dx} = 4\cdot 2(x^{2-1}) = 8x$$

$$\Rightarrow dy = 8x\,dx$$

$(\underset{x}{0},\underset{y}{0})$ $(\underset{x}{1},\underset{y}{4})$

$$= \int_0^1 2x^2(4x^2)dx + 3x(4x^2)8x\,dx$$

$$= \int_0^1 8x^4\,dx + 12x^4\cdot 8\,dx$$

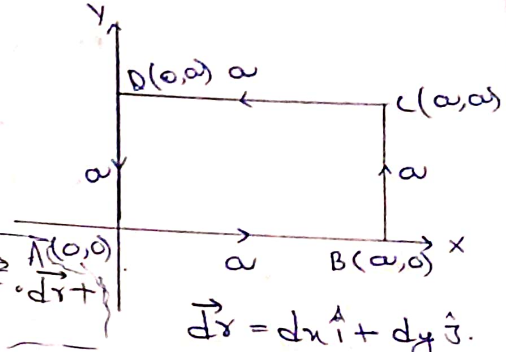
$$= \int_0^1 8x^4\,dx + 96x^4\,dx$$

$$\cancel{= 8\cdot 4\left[x^3\right]_0^1 + 96\cdot 4\left(x^3\right)_0^1}$$

$$\cancel{= 32 + 384}$$

$$= 104\int_0^1 x^4 = \frac{104}{5}\left[x^5\right]_0^1 = \frac{104}{5}\,J.$$

Q. Evaluate $\int_C \vec{F}\,d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\,\hat{j}$ & C is the boundary of the square in the plane $z=0$ & boundary by the lines $x=0$, $y=0$, $x=a$ & $y=a$.



<u>Sol</u>

$$W_{net} = \oint_C \vec{F}\cdot d\vec{r}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}.$$

$$W_{net} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r} \quad \text{—①}$$

<u>Along AB</u> $\boxed{x = a}$ $\boxed{y = 0}$

$$W = \int_0^a \vec{F}\cdot d\vec{r} = \int_0^a (x^2\hat{i} + xy\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_0^a x^2\,dx + xy\,dy$$

$$= \int_0^a x^2\,dx = \frac{1}{2}\left[x^2\right]_0^a = \frac{a^2}{3} \quad \text{—②}$$

Along BC $x = a,\ dx = 0$

$y = a$

$\therefore W = \int_0^a (x^2\hat{i} + xy\hat{j})(dx\hat{i} + dy\hat{j})$

$= \int_0^a \cancel{x^2 dx} + xy\,dy$

$= \int_0^a x\,a\,dy = a\int_0^a y\,dy = a\left[\frac{y^2}{2}\right]_0^a$

$= \frac{a^3}{2}$ — (3)

Along CD $y = a \quad dy = 0$

$x = a$

$W_{CD} = \int_a^0 \vec{F}\cdot\vec{dr} = \int_a^0 x^2 dx + xy\,dy$

$= \int_a^0 x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^2}{3}$ — (4)

Along DA $y = 0 \quad x = 0$

$W_{DA} = \int_0^a x^2 dx + xy\,dy = 0$ — (5)

Putting eq^n (2), (3), (4) & (5) in eq^n (1) we get,

$W_{net} = \frac{a^3}{3} + \frac{a^2}{2} - \frac{a^2}{2} + 0 = \frac{a^3}{3}\hat{j} \quad \frac{a^3}{2}\hat{j}.$

Q. If $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$. Evaluate the line integral $\oint \vec{A}\cdot\vec{dr}$ from (0,0,0) to (1,1,1) along the curve C. $x = t,\ y = t^2$ & $z = t^3$.

Soln $\oint \vec{A}\cdot\vec{dr} = \oint (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$x = t$
$\Rightarrow dx = dt$
$y = t^2$
$\Rightarrow dy = 2t\,dt$
$z = t^3$
$\Rightarrow dz = 3t^2 dt$

$= \oint (3x^2 + 6y)dx - 14yz\,dy + 20xz^2 dz$

$= \oint (3t^2 + 6t^2)dt - 14t^2 t^3 \cdot 2t\,dt + 20\,t\,(t^3)^2 3dt \cdot t^2$

$= \oint 9t^2 dt - 28t^6 + 60t^9 dt$

$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_{(0,0,0)}^{(1,1,1)}$

$= 3(1)^3 - 4(1)^7 + 6(1)^{10}$

$= 3 - 4 + 6$

$= 5$ Joule.

Q. If $\nabla\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (z^3 - 3x^2yz^2)\hat{k}$. find ϕ?

Sol. $\phi = \int_C \vec{F}\,d\vec{s} = \int \nabla\phi\, d\vec{s}$

$= \int (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (z^2 - 3x^2yz^2)dz$

$= \int \underline{y^2 dx} - \underline{2xyz^3 dx} + 3dy + \underline{2xy\,dy} - \underline{x^2z^3 dy} + z^2 dz - \underline{3x^2yz^2 dz}$

$= \int \frac{d}{dx}(y^2x) + \frac{d}{dx}(x^3z^3y) + z^3dz + 3dy$

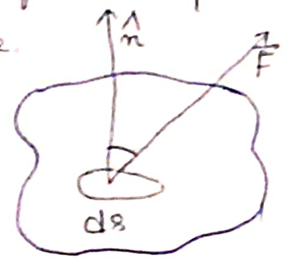
$= y^2x + x^3z^3y + \frac{z^4}{4} + 3y + C$

surface integral :-

Any integral which is to be evaluated over a surface is called a surface integral.

or

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.



Surface integral of F over S

$= \Sigma \vec{F}\cdot\hat{n} = \iint_S (\vec{F}\cdot\hat{n})\,ds$

Note :- $\hat{n} = \frac{\text{grad } F}{|\text{grad } F|}$

1. If R be Projection of 'S' on x-y Plane then

$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

2. If R be Projection of 'S' on yz Plane then

$ds = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

3. If R be Projection of 'S' on xz Plane then

$ds = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

4. Flux $= \iint_S (\vec{F}\cdot\hat{n})\,ds$ where $\vec{F}$ represents the velocity of liquid.

5. If $\iint_S (\vec{F}\cdot\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

Q. Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=1$ in the first octant.

Sol: We know that the surface integral of $\vec{F}$ over S is - $\iint_S \vec{F}\hat{n}\,ds = \iint_S \vec{F}\,\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$$\therefore \hat{n} = \frac{\text{grad } F}{|\text{grad } F|} = \frac{\nabla(x^2+y^2+z^2-1)}{|\nabla x^2+y^2+z^2-1|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$= \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+2y^2+2z^2}}$$

$$= \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\sqrt{\cancel{x^2+y^2+z^2}}}$$

$$= x\hat{i}+y\hat{j}+z\hat{k}.$$

$$\therefore \hat{n}\cdot k = (x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k} = z$$

$$\therefore \vec{F}\,\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= xyz + yzx + xyz$$

$$= 3xyz$$

$$\therefore \iint_S \vec{F}\,\hat{n}\,ds = \iint_R \vec{F}\,\hat{n}\,\frac{dx\,dy}{|\hat{n}\,\hat{k}|}$$

$$= 3\iint_R xy\cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

As x varify from 0 to 1 and y varify from 0 to $\sqrt{1-x^2}$

$$\therefore \iint_S F\hat{n}\,ds = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx\,dy$$

$$= 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x\left(\sqrt{1-x^2}\right)^2 dx$$

$$= \frac{3}{2}\int_0^1 (x-x^3)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{3}{2}\left[\frac{2-1}{4}\right] = \frac{3}{2}\left[\frac{1}{4}\right]$$

$$= \frac{3}{8}$$

Q. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $F = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ as S is the surface of Plane $2x+y+2z=6$ in the first octant.

Sol.
Given, $\text{grad } S = \nabla(2x+y+2z) = 2\hat{i}+\hat{j}+2\hat{k}$

$$\hat{n} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{2^2+1^2+2^2}} = \frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\therefore \hat{n}\cdot\hat{k} = \left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)\hat{k} = \frac{2}{3}$$

$$\therefore \vec{F}\cdot\hat{n} = \left((x+y^2)\hat{i}-2x\hat{j}+2yz\hat{k}\right)\left(\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$$

$$= \frac{1}{3}\left(2(x+y^2)-2x+2\cdot 2yz\right)$$

$$= \frac{1}{3}\left(2(x+y^2)-2x+\cancel{4}^{2}y\left(\frac{6-2x-y}{\cancel{2}}\right)\right)$$

$$= \frac{1}{3}\left(2(x+y^2)-2x+2(6y-2xy-y^2)\right)$$

$$= \frac{1}{3}\left(\cancel{2x}+\cancel{2y^2}-\cancel{2x}+12y-4xy-\cancel{2y^2}\right)$$

$$= \frac{12y-4xy}{3}$$

$$\therefore \iint \vec{F}\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \iint \frac{12y-4xy}{\cancel{3}}\,\frac{\cancel{3}\,dx\,dy}{2}$$

$$= \int_0^3\int_0^{6-2x} 6y-2xy\,dx\,dy$$

$$= 2\int_0^3\int_0^{6-2x} y(3-x)\,dx\,dy$$

$$= 2\int_0^3 (3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\,dx = 2\int_0^3 (3-x)(3-x)^2\,dx$$

$$= 2\int_0^3 (3-x)(9-6x+x^2)\,dx = 2\int_0^3 (27-18x+3x^2-9x+6x^2-x^3)\,dx$$

$$= 2\int_0^3 9x^2-27x+27$$

$$= 2\times 9\int_0^3 x^2-3x+3 = 18\left(\left[\frac{x^3}{3}\right]_0^3 - \frac{3}{2}\left[\frac{x^2}{2}\right]_0^3\right)$$

$$= 18\left(\frac{\cancel{27}^{9}}{\cancel{3}} - \frac{3\times 9}{2}\right)$$

$$= 18\left(\frac{18-27}{2}\right)$$

Volume integral :

Let $\vec{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \iiint_V \vec{F}\, dv$

Q. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, Evaluate $\iiint_V \vec{F}\, dv$ where, V is bounded by the surface.

$x=0,\ y=0,\ x=2,\ y=4,\ z=x^2,\ z=2.$

Soln

$$\iiint_V \vec{F}.dv = \int_0^2\int_0^4\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dx\, dy\, dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\, dz$$

$$= \int_0^2 dx \int_0^4 dx \left[2\frac{z^2}{2}\hat{i} - xz\hat{j} + zy\hat{k}\right]_{x^2}^2$$

$$= \int_0^2 dx \int_0^4 dy \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}\right]$$

$$= \int_0^2 dx \left[4y\hat{i} - 2xy\hat{j} + \frac{2y^2\hat{k}}{2} - x^4y\hat{i} + x^3y\hat{j} - \frac{x^2y^2}{2}\hat{k}\right]_0^4$$

$$= \int_0^2 \left[16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right] dx$$

$$= \left[\cancel{16x\hat{i} - \frac{4x^2}{2}\hat{j} + 16x\hat{k} - \frac{2x^5}{5} + 2\frac{x^4}{4}\hat{j} - \frac{2x^3}{3}\hat{k}}\right]_0^2$$

$$= \left[16x\hat{i} - \frac{8x^2}{2}\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + 4\frac{x^4}{4}\hat{j} - 8\frac{x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{4\times 2^5}{5}\hat{i} + 2^4\hat{j} - \frac{8\times 2^3}{3}$$

$$= 32\hat{i} - \cancel{16\hat{j}} + 32\hat{k} - \frac{4\times 32}{5}\hat{i} + \cancel{16\hat{j}} - \frac{64}{3}\hat{k}$$

$$= \frac{1}{15}\left[32\times 15\hat{i} + 32\times 15\hat{k} - 4\times 3\times 32\hat{i} - 32\times 2\times 5\hat{k}\right]$$

$$= \frac{32}{15}\left[15\hat{i} + 15\hat{k} - 12\hat{i} - 10\hat{k}\right]$$

$$= \frac{32}{15}\left[3\hat{i} + 5\hat{k}\right].$$

Q. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then Evaluate $\iiint_V \nabla\vec{F}\, dv$, where V is bounded by the Plane $x=0,\ y=0,\ z=0$ and $2x + 2y + z = 4$.

Soln

$$\therefore \nabla\vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\left((2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}\right)$$

$$= \frac{\delta}{\delta x}(2x^2 - 3z) - \frac{\delta}{\delta y}(2xy) + \frac{\delta}{\delta z}(4x)$$

$$= 4x - 2x = 2x$$

$z = 4 - 2x - 2y$
$y = 2 - x$
$x = 2$

$$\therefore \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\, dx\, dy\, dz$$

$$= \int_0^2 2x\, dx \int_0^{2-x} dy \int_0^{4-2x-2y} dz$$

$= \int_0^2 2x\,dx \int_0^{2-x} dy\ \boxed{z}_0^{4-2x-2y}$

$= \int_0^2 2x\,dx \int_0^{2-x} (4-2x-2y)\,dy$

$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dy$

$= \int_0^2\int_0^{2-x} (8x-4x^2-4xy)\,dy$

$= \int_0^2 \left[8x-4x^2-4x\left[\frac{y^2}{2}\right]_0^{2-x}\right]$

$= \int_0^2 8x-4x^2-2x(2-x)^2\,dx$

$= \int_0^2 8x-4x^2-2x(4-4x+x^2)\,dx$

$= \int_0^2 (\cancel{8x}-4x^2-\cancel{8x}+8x^2-2x^3)\,dx$

$= \int_0^2 -2x^3+4x^2$

$= 2\int_0^2 x^3-2x^2 = 2\left[-\frac{x^4}{4}+\frac{2x^3}{3}\right]_0^2$

$= 2\left[-\frac{16}{4}+\frac{2\times 8}{3}\right]$

$= 2\left[\frac{-12+16}{3}\right]$

$= 2\left[\frac{4}{3}\right] = \frac{8}{3}$ //

Q. Evaluate $\iiint_V \phi\,dV$, where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Soln

$\therefore \iiint_V \phi\,dV = \iiint_V 45x^2y\,dx\,dy\,dz$

$\Rightarrow 45\int_0^2\int_0^{4-2x}\int_0^{8-4x-2y} x^2y\,dx\,dy\,dz$

$x=2$

$2y+4x=8$

$\Rightarrow y+2x=4$

$\Rightarrow y=4-2x$

$z=8-4x-2y$

$= 45\int_0^2\int_0^{4-2x} x^2y(8-4x-2y)\,dx\,dy$

$= 45\int_0^2\int_0^{4-2x} 8x^2y-4x^3y-2x^2y^2\,dx\,dy$

$= 90\int_0^2\int_0^{4-2x} (4x^2y-2x^3y-x^2y^2)\,dx\,dy$

$= 90\int_0^2 \left[4x^2\frac{y^2}{2}-2x^3\frac{y^2}{2}-x^2\frac{y^3}{3}\right]_0^{4-2x} dx$

STOKES theorem:

$$\oint f \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\,\hat{n}\,ds$$

where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

Proof: Let, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$F = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$.

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$\therefore \oint (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$= \iint_S \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$ $(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})\,ds$

$= \oint f_1dx + f_2dy + f_3dz = \iint_S \hat{i}\left(\frac{\delta f_3}{\delta y} - \frac{\delta f_2}{\delta z}\right) - \hat{j}\left(\frac{\delta f_3}{\delta x} - \frac{\delta}{\delta z}f_1\right) + \left(\frac{\delta f_2}{\delta x} - \frac{\delta f_1}{\delta y}\right)\hat{k}$.

$\Rightarrow \oint f_1dx + f_2dy + f_3dz = \iint_S \left(\frac{\delta f_3}{\delta y} - \frac{\delta f_2}{\delta z}\right)\cos\alpha + \left(\frac{\delta f_1}{\delta z} - \frac{\delta f_3}{\delta x}\right)\cos\beta + \left(\frac{\delta}{\delta x} - \frac{\delta f_1}{\delta y}\right)\cos\gamma$

Let us now prove,

$$\oint f_1 dx = \iint_S \left[\left(\frac{\delta f_1}{\delta z}\cos\beta - \frac{\delta f_1}{\delta y}\cos\gamma\right)\right] ds$$

Let, us eqn $z = g(x, y)$

$\therefore \oint_C f_1(x, y, z)\,dx = \oint_C f_1(x, y, g(x,y))\,dx$

$= -\iint_R \frac{\delta}{\delta y} f_1(x, y, g)\,dx\,dy$

$= -\iint_R \left(\frac{\delta f_1}{\delta y} - \frac{\delta f_1}{\delta z}\frac{\delta g}{\delta y}\right)dx\,dy$

direction cosines

$$\frac{\cos\alpha}{-\frac{\delta g}{\delta y}} = \frac{\cos\beta}{-\frac{\delta g}{\delta x}} = \frac{\cos\gamma}{1}$$

And $dy\,dx = ds\cos\gamma \Rightarrow ds = \frac{dx\,dy}{\cos\gamma}$

$\therefore \iint_S \left[\left(\frac{\delta f_1}{\delta z}\cos\beta - \frac{\delta f_1}{\delta y}\cos\gamma\right)\right] ds =$

$= \iint_R \left(\frac{\delta f_1}{\delta z}\frac{\cos\beta}{\cos\gamma} - \frac{\delta f_1}{\delta y}\frac{\cos\gamma}{\cos\gamma}\right)dx\,dy$

$= \iint_R \frac{\delta f_1}{\delta z}\left(-\frac{\delta g}{\delta y}\right) - \frac{\delta f_1}{\delta y}\Big)\,dx\,dy$

$$= -\iint_R \left(\frac{\delta f_1}{\delta y} + \frac{\delta f_1}{\delta z}\frac{\delta g}{\delta y}\right) dx\,dy$$

from (3) and (4) we get.

Gauss's theorem of divergence:

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv$$

Proof: Let, $F = F_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \iint_S (F_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot\hat{n}\,ds = \iiint_V \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\cdot(f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\,dx\,dy\,dz$$

$$= \iiint_V \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z}\,dx\,dy\,dz \quad \text{—(1)}$$

we required to prove eqn (1)

Let us first evaluate

$$\iiint \frac{\delta f_3}{\delta z}\,dx\,dy\,dz$$

$$\therefore \iiint_S \frac{\delta f_3}{\delta z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\delta f_3}{\delta z}\,dz\right] dx\,dy$$

$$= \iint_R [f_3(x,y,f_2) - f_3(x,y,f_1)]\,dx\,dy \quad \text{—(2)}$$

for the upper part of the surface $dx\,dy = ds_2 \cos\gamma_2 = \hat{n}_2\cdot\hat{k}\,ds_2$

$$\iint_R f_3(x,y,f_2)\,dx\,dy = \iint_{S_2} f_3\,\hat{n}_2\cdot\hat{k}\,ds_2$$

$dx\,dy = ds_1 \cos\gamma_1 = \hat{n}_1\cdot\hat{k}\,ds_1$

~~lower part~~

$$\iint_R f_3(x,y,f_1)\,dx\,dy = -\iint_{S_1} f_3\,\hat{n}_1\cdot\hat{k}\,ds_1$$

Putting eqn ② we have,

$$\iiint_V \frac{\delta f_3}{\delta z} dv = \iint_S f_3 \hat{n}_2 \cdot \hat{k} \, ds^2 + f_3 \hat{n}_1 \cdot \hat{k} \, ds,$$

$$= f_3 \hat{n} \hat{k} \, ds \quad \text{— (iii)}$$

similarly

$$\iiint_V \frac{\delta f_1}{\delta x} dv = \iint_S f_1 \hat{n} \hat{i} \, ds \quad \text{— (iv)}$$

$$\iiint_V \frac{\delta f_2}{\delta x} dv = \iint_S f_2 \hat{n} \hat{j} \, ds \quad \text{(v)}$$

Adding eqn ③ ④ ⑤

$$\iiint \left(\frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z} \right) dv = \iint_S \left(f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k} \right) \hat{n} \, ds$$

$$\Rightarrow \iiint_V \vec{\nabla} \vec{f} \, dv = \iint \vec{F} \hat{n} \, ds$$