Mathematical Physics - 1
Unit - 1

Vector Algebra and Matrices :

⇝ **Scaler Product :-** The scalar Product of two Vector is obtained by mutiplying their magnatudes with the cosine of the angle b/w them.

Scaler Product of two vector $\vec{A}$ and $\vec{B}$ is given by :-
$$\boxed{\vec{A} \cdot \vec{B} = AB\cos\theta}$$
Rem!

where $A = |\vec{A}|$ and $B = |\vec{B}|$
and θ is the angle b/w $\vec{A}$ and $\vec{B}$.

⟶ **ProPerties of Scaler Product :-**

① Scaler Product of two vector may be +ve, -ve or zero.

when, $0 \leq \theta \leq 90°$ then $\cos\theta \rightarrow +ve$
$$\therefore \boxed{\vec{A} \cdot \vec{B} = +ve}$$

when, $90° \leq \theta \leq 180°$ then $\cos\theta \rightarrow -ve$
$$\therefore \boxed{\vec{A} \cdot \vec{B} = -ve.}$$

when, $\theta = 90°$ then $\cos\theta \rightarrow 0$
$$\therefore \boxed{\vec{A} \cdot \vec{B} = 0}$$

So, the condition for two vectors $\vec{A}$ and $\vec{B}$ to be perpendicular $\vec{A}\cdot\vec{B}=0$

(ii) Scaler Product of two vector obeys commutative law:

i.e., $\boxed{\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}}$

(iii) Scaler Product obeys distributive law.

ie $\boxed{\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}}$

(iv) for unit vectors,

$\boxed{\begin{gathered}\hat{i}\cdot\hat{i}=\hat{k}\cdot\hat{k}=\hat{j}\cdot\hat{j}=1\\ \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0\end{gathered}}$

(v) $\vec{A}\cdot\vec{A}=|\vec{A}||\vec{A}|\cos 0^\circ=A^2$

(vi) If $\vec{A}\cdot\vec{B}=0$ then $|\vec{A}|=0$ or $|\vec{B}|=0$ or $\vec{A}\perp\vec{B}$

(vii) If $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$.

$\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$.

Then, $\vec{A}\cdot\vec{B}=(A_x\hat{i}+A_y\hat{j}+A_z\hat{k})(B_x\hat{i}+B_y\hat{j}+B_z\hat{k})$

$\therefore \boxed{\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z}$

→ Physical Significance:

(i) work: $W=\vec{F}\cdot\vec{S}=FS\cos\theta$

(ii) Power: $P=\vec{F}\cdot\vec{v}=Fv\cos\theta$

(iii) Electric flux: $\phi=\vec{E}\cdot\vec{S}=ES\cos\theta$

→ Vector Product or cross Product:

The vector Product of two vector, a and b, is denoted by $a\times b$. Its resultant vector is Perpendicular to a and b.

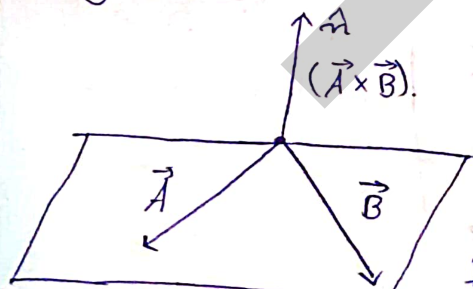
$\therefore \boxed{\vec{A}\times\vec{B}=AB\sin\theta\,\hat{n}}$

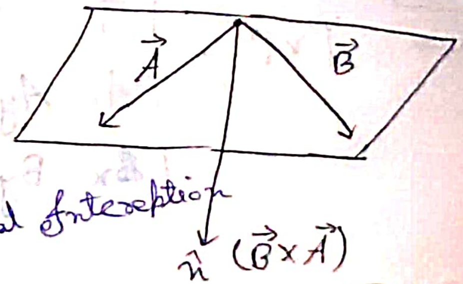
Where, $A=|\vec{A}|$, $B=|\vec{B}|$ and θ is the angle b/w $\vec{A}$ and $\vec{B}$.

$\hat{n}\to$ Unit vector Perpendicular to Plane A and B.

↳ unit vector along $(\vec{A}\times\vec{B})$ Plane.

Direction given by right hand thumb rule or by Maxwell's screw rule.





Physical Interpretation

→ Properties ÷

(I) Vector Product is not commutative.

ie $\boxed{\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \quad (\vec{A} \times \vec{B} = -\vec{B} \times \vec{A})}$

(II) vector Product obey distributed law

ie, $\boxed{\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}}$

(III) If $\vec{A}$ and $\vec{B}$ are collinear (Parallel)

then $\vec{A} \times \vec{B} = 0$

(IV) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}$; $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{k}$.

$\hat{k} \times \hat{j} = -\hat{i}$. $\hat{j} \times \hat{i} = -k$, $\hat{i} \times \hat{k} = -\hat{j}$.

(V) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Then, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

→ Physical ~~Pro~~ Significance ÷

(I) Torque : $\vec{\tau} = \vec{r} \times \vec{F}$

(II) Angular momentum = $\vec{J} = \vec{r} \times \vec{p}$

(III) Angular velocity. $\vec{v} = \vec{\omega} \times \vec{r}$.

(IV) Mgnatic force $\vec{F} = q(\vec{v} \times \vec{B})$

Lorenz law.

(V) Area of Parallalogram = $\vec{A} \times \vec{B} = AB \sin\theta$.

→ Scaler triple Product ÷

Let, the non Zero vectors $\vec{A}$; $\vec{B}$ and $\vec{C}$. then scaler triple Product defined as ÷

$\vec{A} \cdot (\vec{B} \times \vec{C})$ and is denoted by $[\vec{A}\ \vec{B}\ \vec{C}]$

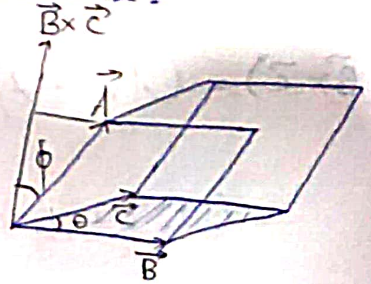
It shows the Volume of Parallelopiped.

→ Geometrical Interpretation ÷

Vector Product of $\vec{B}$ and $\vec{C}$

$= \vec{B} \times \vec{C} = BC \sin\theta\ \hat{n}$

$|\vec{B} \times \vec{C}| = BC \sin\theta$



∴ ~~The def~~

∴ The dot Product of $(\vec{B}\times\vec{C})$ with A –

$= \vec{A}.(\vec{B}\times\vec{C}) = \vec{A}\,(BC\sin\theta)$

$= A\,(BC\sin\theta)\cos\phi$

$= (BC\sin\theta)\,(A\cos\phi)$

Here, $BC\sin\theta$ = Area of base of Parallelopiped and $A\cos\phi$ = hight of Parallelopiped.

∴ $\vec{A}(\vec{B}\times\vec{C})$ = Area of base × hight of Parallelopiped

$\boxed{\vec{A}(\vec{B}\times\vec{C}) = \text{Volume of Parallelopiped}}$ Rem!

→ Scalar ~~Product~~ triple Product in term of Cartesian components:–

We know, $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

∴ $\vec{A}.(\vec{B}\times\vec{C}) = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

$= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\left[\hat{i}(B_yC_z - B_zC_y) - \hat{j}(B_xC_z - B_zC_x) + \hat{k}(B_xC_y - B_yC_x)\right]$

$\Rightarrow \vec{A}.(\vec{B}\times\vec{C}) = A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$.

$$\boxed{\vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}}$$ Rem!

→ Vector triple Product:–

If $\vec{A}, \vec{B}$ and $\vec{C}$ are three non zero Vectors then Vector triple Product is define as $\vec{A}\times(\vec{B}\times\vec{C})$ which is a Vector.

Q. Prove that the cyclic relatn in the Scaler triple Product: $\vec{A}(\vec{B}\times\vec{C}) = \vec{B}(\vec{C}\times\vec{A}) = \vec{C}(\vec{A}\times\vec{B})$

~~Soln~~

Proff:– We know the value of $\vec{A}(\vec{B}\times\vec{C})$ in Cartesian component –

$$A(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Interchanging of two rows of a determinants change its signs, Thus,

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = - \begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\therefore \vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) \quad \text{— (i)}$$

Again,

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = - \begin{vmatrix} C_x & C_y & C_z \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\therefore \vec{A}(\vec{B}\times\vec{C}) = \vec{C}(\vec{A}\times\vec{B}) \quad \text{— (ii)}$$

From eqn (i) and (ii) ⇒

$$\boxed{\vec{A}(\vec{B}\times\vec{C}) = \vec{B}(\vec{C}\times\vec{A}) = \vec{C}(\vec{A}\times\vec{B})}$$ Proved.

Q. If $\vec{A} = 4\hat{i} - 5\hat{j} + 3\hat{k}$, $\vec{B} = 2\hat{i} - 10\hat{j} - 7\hat{k}$, $\vec{C} = 5\hat{i} + 7\hat{j} - 4\hat{k}$ Then find the value of $(\vec{A}\times\vec{B})\cdot\vec{C}$.

Soln: We know that,

$$(\vec{A}\times\vec{B}).\vec{C} = \vec{A}.(\vec{B}\times\vec{C})$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -5 & 3 \\ 2 & -10 & -7 \\ 5 & 7 & -4 \end{vmatrix}$$

$$= 4(40+49) + 5(-8+35) + 3(14+50).$$

$$= 4(89) + 5(27) + 3(64).$$

$$= \cancel{534}\ 356 + 135 + 192$$

$$\cancel{= 861} = 683 \text{ cubic units}$$

534
135
192
861

356
135
192
683

Q. Find the volume of the Parallelopiped whose three edges are represented as, $\vec{a} = 2\hat{i} - 5\hat{j}$, $\vec{b} = 3\hat{j}$, $\vec{c} = 5\hat{j} + 6\hat{k}$.

Soln: We know, volume of Parallelopiped is given by

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -5 & 0 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{vmatrix}$$

$$= 2(18)$$

$$= 36 \text{ cubic units.}$$

→ Vector Triple Product identity :

Q. P.T. $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

Sol. $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

$$= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times\{\hat{i}(B_yC_z - B_zC_y) - \hat{j}(B_xC_z - B_zC_x) + \hat{k}(B_xC_y - B_yC_x)\}$$

$$\cancel{(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ (B_yC_z - B_zC_y) & (B_zC_x - B_xC_z) & (B_xC_y - B_yC_x)\end{vmatrix}$$

$$\cancel{(A_x\hat{i} + A_y\hat{j} + A_z\hat{k})}\{$$

$$= \hat{i}[A_y(B_xC_y - B_yC_x) - A_z(\cancel{B_xC_z - B_zC_x})^{B_zC_x - B_xC_z}] - \hat{j}[A_x(B_xC_y - B_yC_x) - A_z(B_yC_z - B_zC_y)] + \hat{k}[A_x(\cancel{B_xC_z - B_zC_x})^{B_zC_x - B_xC_z} - A_y(B_yC_z - B_zC_y)]$$

$$= \hat{i}[A_yB_xC_y - \cancel{A_yB_yC_x} - A_zB_zC_z + A_zB_zC_x] - \hat{j}[A_xB_xC_y - A_xB_yC_x - A_zB_yC_z + A_zB_zC_y] + \hat{k}[A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y]$$

$$= \hat{i}(A_yB_xC_y + A_zB_xC_z) - \hat{i}(A_yB_yC_x + A_zB_zC_x) -$$

$$\hat{j}(A_xB_xC_y + A_zB_zC_y) + \hat{j}(A_xB_yC_x + A_zB_yC_z) +$$

$$\cancel{\hat{k}(A_xB_xC_z + A_yB_zC_y) - \hat{k}(A_xB_zC_x + A_yB_yC_z)}$$

$$\hat{k}(A_xB_zC_x + A_yB_zC_y) - \hat{k}(A_xB_xC_z - A_yB_yC_z).$$

$$\cancel{= \hat{i}B_x(A_yC_y + A_zC_z) - \hat{i}C_x}$$

$$= [\hat{i}B_x(A_yC_y + A_zC_z + A_xC_x) + \hat{j}C_y(A_xB_x + A_zB_z + A_yB_y)$$

$$+ \hat{k}B_z(A_xC_x + A_yC_y + A_zC_z)] - [\hat{i}C_x(A_yB_y + A_zB_z + A_xC_x)$$

$$+ \hat{j}B_y(A_xC_x + A_zC_z + A_yC_y) + \hat{k}C_z(A_xB_x + A_yB_y + A_zB_z)$$

$$= [\cancel{(\hat{i}B_x + \hat{j}C_y + \hat{k}B_z)}(A_yC_y + A_zC_z + A_xC_x)] -$$

$$[(\hat{i}C_x + \hat{j}B_y + \hat{k}C_z)(A_xB_x - A_yB_y + A_zB_z)$$

$$= [\hat{i}B_x(A_yC_y + A_zC_z + A_xC_z) + \hat{j}B_y(A_xC_x + A_zC_z + A_yC_y)$$

$$+ \hat{k}B_z(A_xC_x + A_yC_y + A_zC_z)] - [\hat{i}C_x(A_yB_y + A_zB_z + A_xC_x)$$

$$+ \hat{j}C_y(A_xB_x + A_zB_z + A_yB_y) + \hat{k}C_z(A_xB_x + A_yB_y + A_zB_z)$$

$$= [(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z)(A_xC_x + A_yC_y + A_zC_z)] -$$

$$[(\hat{i}C_x + \hat{j}C_y + \hat{k}C_z)(A_xB_x + A_yB_y + A_zB_z)]$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Proved

Note:-

Identity: $\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{B} & \vec{C} \\ \vec{A}\cdot\vec{B} & \vec{A}\cdot\vec{C} \end{vmatrix}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}).$$

Cyclic relation

Q. P.T. $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$

Soln

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{B} & \vec{C} \\ \vec{A}\cdot\vec{B} & \vec{A}\cdot\vec{C} \end{vmatrix}$$

$$= \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \quad \text{—(1)}$$

$$\vec{B}\times(\vec{C}\times\vec{A}) = \begin{vmatrix} \vec{C} & \vec{A} \\ \vec{B}\cdot\vec{C} & \vec{B}\cdot\vec{A} \end{vmatrix}$$

$$= \vec{C}(\vec{B}\cdot\vec{A}) - \vec{A}(\vec{B}\cdot\vec{C}) \quad \text{—(2)}$$

And $\vec{C}\times(\vec{A}\times\vec{B}) = \begin{vmatrix} \vec{A} & \vec{B} \\ \vec{C}\cdot\vec{A} & \vec{C}\cdot\vec{B} \end{vmatrix}$

$$= \vec{A}(\vec{B}\cdot\vec{C}) - \vec{B}(\vec{A}\cdot\vec{C}) \quad \text{—(3)}$$

Adding eqn (1), (2), (3) ⇒

$$\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}(\vec{A}\times\vec{B}) = 0$$

Proved

Q. P.T. $\hat{i}\times(\vec{v}\times\hat{i}) + \hat{j}\times(\vec{v}\times\hat{j}) + \hat{k}(\vec{v}\times\hat{k}) = 2\vec{v}$

Soln

W/K, $\vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$

and $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.

Now we take,

$$\hat{i}\times(\vec{v}\times\hat{i}) = \vec{v}(\hat{i}\cdot\hat{i}) - \hat{i}(\vec{v}\cdot\hat{i})$$

$$= \vec{v} - \hat{i}\{\hat{i}(\hat{i}v_x + \hat{j}v_y + \hat{k}v_z)\}$$

$$= \vec{v} - v_x \quad \text{—(1)}$$

Similarly $\hat{j}\times(\vec{v}\times\hat{j}) = \vec{v}(\hat{j}\cdot\hat{j}) - \hat{j}(\hat{j}\cdot\vec{v})$

$$= \vec{v} - \hat{j}\{\hat{j}(\hat{i}v_x + \hat{j}v_y + \hat{k}v_z)\}$$

$$= \vec{v} - \hat{j}v_y \quad \text{—(2)}$$

Similarly, $\hat{k}(\vec{v}\times\hat{k}) = \vec{v}(\hat{k}\cdot\hat{k}) - \hat{k}(\hat{k}\cdot\vec{v})$

$$= \vec{v} - \hat{k}\{\hat{k}(\hat{i}v_x + \hat{j}v_y + \hat{k}v_z)\}$$

$$= \vec{v} - \hat{k}v_z \quad \text{—(3)}$$

Adding eqn ①, ②, ③ ⇒

$$\hat{i}\times(\hat{v}\times\hat{i}) + \hat{j}(\hat{v}\times\hat{j}) + \hat{k}(\hat{v}\times\hat{k}) = \vec{v} - \hat{i}v_x + \vec{v} - \hat{j}v_y + \vec{v} - \hat{k}v_z$$

$$= 3\vec{v} - (\hat{i}v_x + \hat{j}v_y + \hat{k}v_z)$$

$$= 3\vec{v} - \vec{v}$$

$$= 2v$$

Proved

Q. P.T. $[\vec{a}+\vec{b} \cdot \vec{b}+\vec{c} \cdot \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

Soln L.H.S

$$[\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+\vec{a}] = (\vec{a}+\vec{b})[(\vec{b}+\vec{c})\cdot(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b})[(\vec{b}\vec{c} + \vec{b}\vec{a}) + (\vec{c}\vec{c} + \vec{c}\vec{a})]$$

$$= [\vec{a}\vec{b}\vec{c} + \vec{b}\vec{b}\vec{c} + \vec{a}\vec{b}\vec{a} + \vec{b}\vec{a}\vec{a} + \vec{b}\vec{b}\vec{a} + \vec{a}\vec{c}\vec{c} + \vec{a}\vec{c}\vec{a} + \vec{b}\vec{c}\vec{c} + \vec{c}\vec{a}\vec{b}]$$

$$= [\vec{a}\vec{b}\vec{c} + 0+0+0+0+0+0+0+\vec{c}\vec{a}\vec{b}]$$

$$= [\vec{a}\vec{b}\vec{c} + \vec{a}\vec{b}\vec{c}]$$

$$= 2[\vec{a}\,\vec{b}\,\vec{c}]$$

RHS

Proved

Q. P.T. $[\vec{a}\times\vec{b} \cdot \vec{b}\times\vec{c} \cdot \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

Soln ~~$\vec{b}\times\vec{c}$~~ =

L.H.S. $[\vec{a}\times\vec{b} \cdot \vec{b}\times\vec{c} \cdot \vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$ — ①

Let, $\vec{b}\times\vec{c} = \vec{d}$ then,

$$(\vec{b}\times\vec{c})(\vec{c}\times\vec{a}) = \vec{d}\times(\vec{c}\times\vec{a})$$

$$= \vec{c}(\vec{d}\,\vec{a}) - \vec{a}(\vec{c}\,\vec{d})$$

$$= (\vec{b}\times\vec{c}\cdot\vec{a})\vec{c} - (\vec{b}\times\vec{c}\cdot\vec{c})a$$

$$= (\vec{a}\cdot\vec{b}\times\vec{c})\vec{c} - (0)a$$

$$= [\vec{a}\cdot\vec{b}\times\vec{c}]\vec{c}$$

$$= [\vec{a}\cdot\vec{b}\,\vec{c}]\vec{c}$$

$$\boxed{\begin{aligned}&\vec{A}\times(\vec{B}\times\vec{c})\\ &= \vec{B}(\vec{A}\cdot\vec{c}) - \vec{c}(\vec{A}\cdot\vec{B})\\ &= (\vec{A}\,\vec{c})\vec{B} - (\vec{A}\cdot\vec{B})c.\end{aligned}}$$

from eqn①

$$\therefore \vec{a}\times\vec{b}\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$$

$$= (\vec{a}\times\vec{b})\cdot[\vec{a}\cdot\vec{b}\times\vec{c}]\vec{c}$$

$$= (\vec{a}\,\vec{b})\cdot\vec{c}\,[\vec{a}\cdot\vec{b}\times\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]\cdot[\vec{a}\cdot\vec{b}\times\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}][\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]^2$$

RHS.

Proved

→ **field:**

If at any Point in a region of space, any Physical quantity can be expressed as the Position coordinate of the Point, then this region of space is called field.

Since there are two types of Physical quantities Vector and scalar. Therefore two types of field are Possible:

① Scalar field ② Vector field.

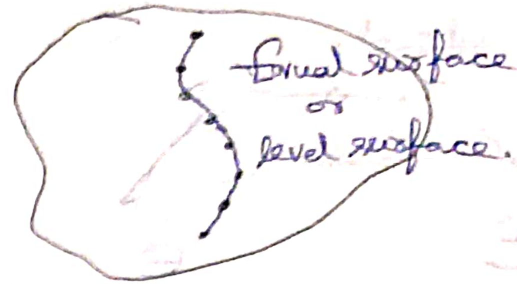
→→ **Scalar field:**

If any Physical quantity at any Point in a region of space can be expressed as a scaler functⁿ of the Position coordinate of that Point, then this region of space is called a scaler field.

In a scaler field, the Points on which the magnitude of Physical quantity is equal can connected by a surface, which is called equal surface or level surface.

Therefore, the value of Physical quantity will be the same at all Points of equal surface.

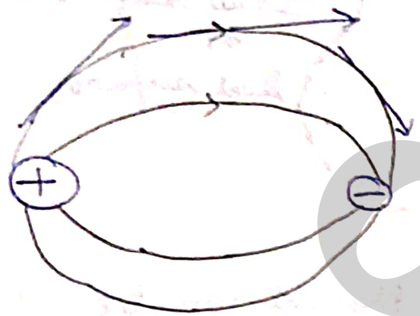
Example: Tempʳ, Electric Potential, density etc.



→→ **Vector field:**

If any Physical quantity at any Point in a region of space can be expressed as a vector functⁿ of the Position coordinate of the Point, this region of space is called vector field.

Therefore, to express a Physical quantity in a vector field, both magnitude and direction are required. If we move Point by Point in the vector field, then a curve is usually obtained, which is called flux line or vector line or line of flow. The tangent at any Point on this flow line will represent the direction of the Physical quantity at that Point and the number of flux lines Passing through the ~~Eeokock~~ unit area at that Point will represent the magnitude of the Physical quantity at that Point. Ex: Distribution of electric field $E(x,y,z)$ in a region surrounded by a charge object.



Different types of matrices :

(I) column matrix : A matrix is said to be a column matrix if it has only one column.

for example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4x

In general, $A = [a_{ij}]_{m\times 1}$ is a column matrix of order $m\times 1$.

(II) Row matrix : A matrix is said to be a row matrix if it has only one row.

for example, $B = \begin{bmatrix} -1/2 & \sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matri

In general, $B = [b_{ij}]_{1\times n}$ is a row matrix of order $1\times n$.

(III) Square matrix : A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus an $m\times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'

for example, $A = \begin{bmatrix} 3 & -1 & 0 \\ 3/2 & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

in General, $A = [a_{ij}]_{m\times m}$ is a square matrix of order m.

(iv) Rectangular matrix ÷ Any matrix A is said to be rectangular matrix if the number of rows not equal to the no. of colums.

ie, $R \neq C$

Ex. $A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 8 & 5 \end{bmatrix}$, $X = \begin{bmatrix} \alpha, \beta, \mu, \delta, \phi, \lambda \end{bmatrix}_{1\times 6}$

Q. The matrix $A = (a_{ij})_{m\times n}$ is said to be squre matrix if -

(i) $m = n$ (ii) $m > n$ (iii) $m < n$ (iv) N.O.T.

Ans: (i) $m = n$.

(v) Singular matrix ÷ Any squre A is said to be singular matrix if $|A| = 0$.

Ex. $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}_{2\times 2}$

$|A| = \begin{vmatrix} 1 & i \\ -i & 1 \end{vmatrix} = 1 + i^2 = 1 - 1 = 0$; clearly given matrix is singular.

Q. Show that the matrix $A = \begin{bmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{bmatrix}_{3\times 3}$

Sol $A = \begin{bmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{bmatrix}$

$\Rightarrow |A| = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$= \begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix}$ $C_2 \to C_1 + C_2$

$= (a+b+c) \begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$ C_2 and C_3 are identical

$= 0$; Hence the given matrix are is ~~identical~~ singular.

(vi) NON singular matrix ÷ Any square matrix 'A' is said to be non-singular matrix if there determinants not equal to zero.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}_{2\times 2}$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix}$

$= 7 - 4 = 3 \neq 0$ clearly the matrix is non singular.

(vii) zero matrix or null matrix ÷ Any matrix either it is square or rectangular whose all the elements are zero is called zero-matrix or null matrix.

Ex. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$.

(VIII) Unit matrix:- Any square matrix is said to be unit matrix whose all the elements along the principle diagonal elements are zero is called unit matrix.

unit matrix is denoted by 'I'.

Ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

(IX) Diagonal matrix:- Any square matrix 'A' is said to be diagonal matrix if all the elements along the principal diagonal are unequal and the non-diagonal elements are zero.

Ex. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$ $B = \begin{bmatrix} x+1 & 0 & 0 \\ 0 & x^2+2 & 0 \\ 0 & 0 & x^3+3 \end{bmatrix}_{3\times 3}$

(X) Scalar matrix:- Any square matrix 'A' is said to be scalar matrix whose all the elements along the principle diagonal are equal and non-diagonal elements are zero are called scalar matrix.

Ex. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$ P.D

⇝ Symmetric matrices:- Any square matrix A is said to be symmetric matrix if $A = A'$

$A = A'$

Ex. $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}_{2\times 2}$ $A' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}_{2\times 2} = A.$

Let, $A = a_{ij}$

$A' = a_{ji}$

for symmetric matrix,

$\Rightarrow A = A'$

$\Rightarrow a_{ij} = a_{ji}$

$\Rightarrow \boxed{\frac{a_{ij}}{a_{ji}} = 1}$

⇝ (Skew) Antisymmetric matrices:- Any square matrix 'A' is said to be Antisymmetric matrix.

if, $A = -A'$

$\Rightarrow A + A' = 0$

Ex. $A = \begin{bmatrix} 0 & e & -f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}_{3\times 3}$ $A' = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}_{3\times 3}$

Now, $A + A' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$ is a zero matrix.

$\Rightarrow A + A' = 0$

Theorem- If any square matrix then show that $A+A'$ is symmetric and $A-A'$ is Antisymmetric.

Proff:-

Let, $X = A + A'$

for Symmetric, $X' = X$

$\therefore X = A + A'$

$\Rightarrow X' = (A+A')'$

$\Rightarrow X' = A' + (A')'$

$\Rightarrow X' = A' + A$

$\Rightarrow X' = A + A'$

$\Rightarrow X' = X$

$\therefore$ clearly, $X = A + A'$ is symmetric.

Let, $X = A - A'$

for antisymmetric, $Y + Y' = 0$

$\Rightarrow Y' = (A - A')'$

$\Rightarrow Y' = A' - (A')'$

$\Rightarrow Y' = A' - A$

$\Rightarrow Y' = -(A - A')$

$\Rightarrow Y' = -Y$

$\Rightarrow Y + Y' = 0$

clearly

$\Rightarrow$ Two express the square matrix 'A' as the sum of symmetric and antisymmetric matric.

$$\therefore A = \frac{1}{2}\{\underbrace{(A+A')}_{\text{Symmetric}} + \underbrace{(A-A')}_{\text{Antisymmetric}}\}$$

Q. Two express the square matrix $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of symmetric and antisymmetric matrix.

Soln Let, $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ R. $A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

$$\therefore A = \frac{1}{2}\{(A+A') + (A-A')\}$$

$$\Rightarrow \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right\} + \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\left\{\underbrace{\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}}_{\text{Antisymmetric}}\right\}$$

$\Rightarrow$ Hermitian matrix:- A square matrix that is equal to the transpose of its conjugate matrix.

i.e., $(\bar{A})' = A$

Q. find if the matrix $A = \begin{bmatrix} -99 & 6+7i \\ 6-7i & 0 \end{bmatrix}$ is a hermitian matrix.

Soln Given matrix, $A = \begin{bmatrix} -99 & 6+7i \\ 6-7i & 0 \end{bmatrix}$

first we find the conjugate of A,

$$\bar{A} = \begin{bmatrix} -99 & 6-7i \\ 6+7i & 0 \end{bmatrix}$$

$\therefore (\bar{A})' = \begin{bmatrix} -99 & 6+7i \\ 6-7i & 0 \end{bmatrix}$

$\Rightarrow (\bar{A})' = A$

Thus the given matrix A is a hermition matrix.

→ Properties of hermitian matrix :-

1. The elements of the Principal diagonal of a hermition matrix are always real numbers.

2. The non-diagonal elements of the hermitian matrix can be complex numbers.

3. If A is a hermitian matrix and K is any real Scalar, then KA is also a hermitial matrix. This is because $(KA)' = KA' = KA$, if K is a real number.

4. The sum of two hermition metrices is again a hermitian matrix.

5. The Product of two hermitian matrices is hermitian.

6. If A and B are square matrices, then $(AB)' = B'A'$. If A and B are hermitian, then $(AB)' = BA$.

7. The inverse of a hermition matrix is a hermitian matrix.

8. The determinant of a hermitian matrix is always real.

9. The conjugate of a hermitian matrix is also a hermitian matrix.

10. Any square matrix can be represented as $A+iB$, Where A and B are hermitian matrices.

11. Given a square matrix A, if $A' = -A$, then A is called the Skew-hermitian matrix.

12. If A is a hermitian matrix, then A^n is also hermitian for all Positive integers n.

13. The trace of a hermitian matrix is always real.

Q. If K is complex number and A be a hermitian matrix. Will KA be hermitian?

Soln Given, A is a hermitian matrix, $A' = A$ and K is any complex number.

Now, $(KA)' = \bar{K}A' = \bar{K}A \neq KA$.

Since k is a complex number, and it is not equal to its ~~co~~ conjugate.

Hence, $(kA)' \neq kA$.

Thus, kA is not a hermitian matrix.

⇝ Inverse of matrices :- The inverse of a matrix is another matrix that, when ~~matrix~~ multiplied by the given matrix, yields the multiplicative identity.

for a matrix A, its inverse is A^{-1} and $A \cdot A^{-1} = I$, where I ~~be~~ is denoted as the identity matrix.

Q. find the adjoint (adj) and inverse of matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}_{2\times 2}$$

Sol. let, $A = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}_{2\times 2}$

Step I, Determinant test

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7$$

clearly, $|A| \neq 0$

then, A^{-1} exist.

Step II :-

$C_{11} = 5$; $C_{12} = -1$

$C_{21} = -3$; $C_{22} = 2$

let, $B = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$

$$\text{Adj } A = B' = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}A}{|A|} = \frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5/7 & -3/7 \\ -1/7 & 2/7 \end{bmatrix}$$

Q. find the ~~an~~ order and inverse of the matrix -

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}.$$

Sol. let, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}.$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$ Expending along R_1

$= 1(10-0) = 10$; clearly A^{-1} exist

$C_{11} = (-1)^2 M_{11} = 1 \times \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10$

$C_{12} = (-1)^3 M_{12} = -1 \times \begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0$

$C_{13} = (-1)^4 M_{13} = 1 \times \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{21} = (-1)^3 M_{12} = -1 \times \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10$

$C_{22} = (-1)^4 M_{22} = 1 \times \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5$

$C_{23} = (-1)^5 M_{23} = -1 \times \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^4 M_{31} = 1 \times \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$

$C_{32} = (-1)^5 M_{32} = -1 \times \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$

$C_{33} = (-1)^6 M_{33} = 1 \times \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$

$$B = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}$$

$$Adj A = B' = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{Adj A}{|A|} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1/5 \\ 0 & 1/2 & -2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Important results :-

1. If A be any square matrix, then ~~$A(AdjA) = AdJ$~~

$A(AdjA) = (AdjA)A$

$= |A| \cdot I$.

2. If A and B are two square matrix of same order then, $|A \cdot B| = |A| \cdot |B|$.

3. If A be any square matrix order n, then

$|Adj A| = |A|^{n-1}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 0 \\ -1 & 2 & 5 \end{bmatrix}_{3 \times 3} \qquad n = 3$$

$|Adj A| = |A|^{3-1} = |A|^2$

4. If A and B be square matrix of order n then,

$A \cdot I = I \cdot A = A$.

$B \cdot I = I \cdot B = B$

$A^{-1} I = I A^{-1} = A^{-1}$

$B^{-1} I = I B^{-1} = B^{-1}$

$A A^{-1} = A^{-1} A = I$

$B B^{-1} = B^{-1} B = I$.

Q. If $A = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$ Show that $A^2 - 5A + 7I_2 = 0$

Hence find A^{-1}.

Sol: Step 1: $A^2 = A \times A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}_{2\times 2}$

Step 2: $-5A = -5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix}$

Step 3: $7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Hence, $A^2 - 5A + 7I_2$

$= A^2 + (-5A) + 7I_2$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\therefore A^2 - 5A + 7I = 0$

Pre multiplying both sides by A^{-1}

$A^{-1}(A^2 - 5A + 7I) = A^{-1} \cdot 0$

$\Rightarrow A^{-1}A^2 - 5A^{-1}A + 7A^{-1} - I = 0$

$\Rightarrow A^{-1} \cdot A \cdot A - 5A^{-1}A + 7A^{-1}I = 0$

$\Rightarrow IA - 5I + 7A^{-1} = 0$

$\Rightarrow A - 5I + 7A^{-1} = 0$

$7A^{-1} = 5I - A = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix}$

$\Rightarrow 7A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ 1/7 & 3/7 \end{bmatrix}_{2\times 2}$

⇝ Transpose of matrices: The transpose of a matrix is found by interchanging its rows into columns or columns into rows.

The transpose of the matrix is denoted by using the letter "T" is the superscript of the given matrix.

Ex: $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3}$ $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3\times 2}$

Q. find the transpose of the given matrix.

$M = \begin{bmatrix} 2 & -9 & 3 \\ 13 & 11 & -17 \\ 3 & c & 15 \\ 4 & 13 & 1 \end{bmatrix}$

~~Sol~~ Given a matrix of the order 4x3

the transpose of a matrix is given by interchanging rows and columns

$$M^T = \begin{bmatrix} 2 & 13 & 3 & 4 \\ -7 & 11 & 6 & 13 \\ 3 & -17 & 15 & 1 \end{bmatrix}$$

Q. find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ by elementary transformation

Sol: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ using elementary row operation

ie, $A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \cdot A \quad R_2 \to 2R_1 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot A ; \quad R_2 \to -1 \times R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \cdot A ; \quad R_1 \to R_1 - 2R_2$$

$$\Rightarrow I = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \cdot A$$

$\Rightarrow$

$$\Rightarrow A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Q. find the inverse of the matrix

$\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ using elementary row operation

Sol: $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ using elementary row operation

ie, $A = IA$.

$$\Rightarrow \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A ; \quad R_1 \to R_3 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & -5 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 2 \\ -3 & 0 & 2 \end{bmatrix} \cdot A ; \quad \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2/5 & 1/5 & -2/5 \\ 3/5 & 0 & -2/5 \end{bmatrix} \cdot A ; \quad \begin{matrix} R_2 \to -1/5\, R_2 \\ R_3 \to -1/5\, R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 3/5 & 0 & -2/5 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} \cdot A \quad R_3 \to R_2 + R_3$$

$$\Rightarrow I = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} . A$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix}$$ //

⇝ **Solution of simultaneous linear eqn's:**

Q. Solve the system of linear eqn's:

$$\left.\begin{aligned} 2x + 3y &= 8 \\ 3x + 7y &= 17 \end{aligned}\right\} \text{By matrix method.}$$

Sol: Let, $A = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $B = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$

Such that $A . X = B$

$\Rightarrow X = A^{-1} . B$ — ①

$|A| = \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = |4 - 9| = 5$; A^{-1} is exist.

$Adj\,A = \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}$ $\therefore A^{-1} = \dfrac{Adj\,A}{|A|}$

$= \dfrac{1}{5} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix}$

from eqn ①

$\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{5} \begin{bmatrix} 7 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 17 \end{bmatrix}$

$= \dfrac{1}{5} \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\therefore x = 1$; $y = 2$.

Q. Solve the system of linear eqn's.

$x - y + z = 4$

$2x + y - 3z = 0$ By matrix method.

$x + y + z = 2$

Sol: Let, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

Such that, $AX = B$

$\Rightarrow X = A^{-1} B$ — ①

$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$ Expending along R_1

$= 1 \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= 1(1+3) + 1(2+3) + 1(2-1)$

$= 4 + 5 + 1$

$= 10$; clearly A^{-1} exist.

$C_{11} = (-1)^2 M_{11} = 1 \times \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4$

$C_{12} = (-1)^3 M_{12} = -1 \times \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$

$C_{13} = (-1)^4 M_{13} = 1 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$

$C_{21} = (-1)^3 M_{21} = -1 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2$

$C_{22} = (-1)^4 M_{22} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$C_{23} = (-1)^5 M_{23} = -1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2$

$C_{31} = (-1)^4 M_{31} = 1 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2$

$C_{32} = (-1)^5 M_{32} = -1 \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5.$

$C_{33} = (-1)^6 \cdot M_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3.$

$$C = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\text{AdJ } A = C' = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{AdJ} A}{|A|} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

from eqn ①

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore x = 2;\ y = -1,\ z = 1$

⇝ Eigen value of matrices ÷ The special set of Scalar value that is associated with the set of linear equations most Probably in the matrix equations.

The basic eqn is ÷ $Ax = \lambda x$.

The number or scalar value 'λ' is an eigenvalue of A.

⇒ Eigenvalue of Square of matrix ÷

Suppose, $A_{n \times n}$ is a square matrix, then $\boxed{A - \lambda I}$ is called an Eigen or characteristic matrix, which is an ~~iten~~ indefinite or matrix can be written as. $|A-\lambda I|$ and $|A-\lambda I|=0$ is Eigen equation or characteristics equation, where I is the identity matrix.

Q. find the eigenvalues of the 2×2 matrix

$$A=\begin{bmatrix}2 & 1\\ 4 & 5\end{bmatrix}$$

Sol Given. $A=\begin{bmatrix}2 & 1\\ 4 & 5\end{bmatrix}$

Using the characteristic eqn,

let, $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

be the 2×2 identity matrix.

$|A-\lambda I|=0$

$\Rightarrow \left|\begin{bmatrix}2 & 1\\ 4 & 5\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|=0$

$\Rightarrow \left|\begin{bmatrix}2 & 1\\ 4 & 5\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right|=0$

$\Rightarrow \begin{vmatrix}2-\lambda & 1\\ 4 & 5-\lambda\end{vmatrix}=0$

$\Rightarrow \cancel{-\lambda(4-\lambda)-(-2)(9)=0}$

$\Rightarrow \cancel{-4\lambda+\lambda^2+6=0}$

$\Rightarrow \cancel{\lambda^2-4\lambda+6=0}$

$\Rightarrow \begin{vmatrix}2-\lambda & 1\\ 4 & 5-\lambda\end{vmatrix}=0$

$\Rightarrow (2-\lambda)(5-\lambda)-4=0$ $\quad \Rightarrow \cancel{(2-\lambda)(5-\lambda)=4}$

$\cancel{2-\lambda=4}$

$\Rightarrow -\lambda=4-2$

$\Rightarrow 10-2\lambda-5\lambda+\lambda^2-4=0$

$\Rightarrow \lambda^2-7\lambda+6=0$

$\Rightarrow \lambda^2-(6+1)\lambda+6=0$

$\Rightarrow \lambda^2-6\lambda-\lambda+6=0$

$\Rightarrow \lambda(\lambda-6)-1(\lambda-6)=0$

$\Rightarrow (\lambda-1)(\lambda-6)=0$

Either, $\lambda-1=0$ or, $\lambda-6=0$

$\Rightarrow \lambda=1$ $\quad \Rightarrow \lambda=6.$

Hence the required eigen values ~~of~~ are 6 and 1.

Q. Calculate the eigenvalue ~~eqn~~ for the following matrix -

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 2\\ 2 & 0 & 0\end{bmatrix}$$

~~Sol~~ ~~Either~~ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$

using the charecteristic eqn:

let, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

be the 3x3 identical metrix.

$|A - \lambda I| = 0$

$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$

$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)[(-1-\lambda)(-\lambda)] = 0$

$\cancel{\Rightarrow (1-\lambda)[(\lambda+\lambda^2)] = 0}$

$\cancel{\Rightarrow \lambda + \lambda^2 - \lambda^2 - \lambda^3 = 0}$

$\cancel{\Rightarrow \lambda^3 - \lambda = 0}$

$\Rightarrow -\lambda(-1-\lambda)(1-\lambda) = 0$

~~Either, $(-1-\lambda) = 0$~~

$\Rightarrow \lambda(\lambda+1)(1-\lambda) = 0$

Either, $\lambda = 0$ or, $(\lambda+1)(1-\lambda) = 0$

$\cancel{\Rightarrow \lambda + 1 - \lambda^2 - \lambda}$

$\Rightarrow \lambda - \lambda^2 + 1 - \lambda = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm\sqrt{1} = \pm 1$

$\therefore$ The estimate eigenvalue are $\lambda = 0, 1, -1$

Q. find the Eigenvalue for the matrix.

$A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$

Soln Given, $A = \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix}$

using the charecteredic eqn:

let, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

be the 3x3 identical ~~eqn:~~ matrix.

$|A - \lambda I| = 0$

$\Rightarrow \left| \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$

$$\Rightarrow \left| \begin{bmatrix} 4 & 6 & 10 \\ 3 & 10 & 13 \\ -2 & -6 & -8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 6 & 10 \\ 3 & 10-\lambda & 13 \\ -2 & -6 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 4-\lambda\left[(10-\lambda)(-8-\lambda)-(13)(-6)\right] - 6\left[(3)(-8-\lambda)-(13)(-2)\right]$$
$$+10\left[3(-6)-(10-\lambda)(-2)\right] = 0$$

Now, we can take 1st part from the above eqn

$$4-\lambda\left[(10-\lambda)(-8-\lambda)-(13)(-6)\right]$$

$$\Rightarrow (4-\lambda)\left[-80-10\lambda+8\lambda+\lambda^2+78\right]$$

$$\Rightarrow (4-\lambda)\left[\lambda^2-2\lambda-2\right] \quad \text{— (I)}$$

Now, we can take 2nd part from the above eqn,

$$-6\left[(3)(-8-\lambda)-(13)(-2)\right]$$

$$\Rightarrow -6\left[-24-3\lambda+26\right]$$

$$\Rightarrow 18\lambda-12 \quad \text{— (II)}$$

Now, we can take 3rd part from the above eqn.

$$10\left[-18+(20-2\lambda)\right]$$

$$= 10(-2\lambda+2)$$

$$= -20\lambda+20 \quad \text{— (III)}$$

Now Simplifying eqn (I)

$$(4-\lambda)(\lambda^2-2\lambda-2)$$

$$= 4\lambda^2-8\lambda-8-\lambda^3+2\lambda^2+2\lambda$$

$$= 6\lambda^2-\lambda^3-6\lambda-8 \quad \text{— (IV)}$$

Now adding (II) + (III) + (IV)

$$18\lambda-\cancel{12}-20\lambda+\cancel{20}+6\lambda^2-\lambda^3-6\lambda-\cancel{8}=0$$

$$\Rightarrow -\lambda^3+6\lambda^2-8\lambda=0$$

$$\Rightarrow \lambda(-\lambda^2+6\lambda-8)=0$$

Either, $\lambda=0$ or $-\lambda^2+6\lambda-8=0$

$$\Rightarrow \lambda^2-6\lambda+8=0$$

$$\Rightarrow \lambda^2-(4+2)\lambda+8=0$$

$$\Rightarrow \lambda^2-4\lambda-2\lambda+8=0$$

$$\Rightarrow \lambda(\lambda-4)-2(\lambda-4)=0$$

$$\Rightarrow (\lambda-2)(\lambda-4)=0$$

2 | 18
2 | 4
  | 2

Either, $\lambda - 2 = 0$ $\Rightarrow \lambda = 4$

$\Rightarrow \lambda = 2$

$\therefore$ Now, the estimate eigenvalues are $\lambda = 0, 2, 4$

Eigenvectors: Eigenvectors of a square matrix are non-zero vectors, which, when multiplied by the square matrix, result is the scalar multiple of the vectors.

If A be an $n \times n$ matrix and λ be the Eigenvalues associated with it. Then, Eigenvalue defined by the following reaction:

$$AV = \lambda V$$

If I is the identity matrix of same order as A, then,

$$(A - \lambda I) V = 0$$

Here V is known as the eigen ~~value~~ vector belonging to each eigen value and is written as:

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

Q. Find the Eigen value for the following matrix:

$$A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$$

Soln: Given, $A = \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix}$

using the characteristic eqn,

Let, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

be the 2×2 identical matrix.

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ -4 & -7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(-7-\lambda) + 16 = 0$$

$$\Rightarrow -7 - \lambda + 7\lambda + \lambda^2 + 16 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$\lambda^2 + 3\lambda + 3\lambda + 9 = 0$$

$$\Rightarrow \lambda(\lambda+3) + 3(\lambda+3) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3) = 0$$

Either, $\lambda + 3 = 0$

$\Rightarrow \lambda = -3$

or, $\lambda + 3 = 0$

$\Rightarrow \lambda = -3$

∴ The estimate eigen values is = -3

using eigen vector even,

$$AX = -3X$$

$$(A + 3I)X = O$$

$$\Rightarrow \left(\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 1 & 4 \\ -4 & -7 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \cancel{4x + 4y = 0 \,/\, -4x + 4y = 0}$

$\Rightarrow \cancel{x + y = 0}$ $\quad \Rightarrow \cancel{-x - y = 0}$

$\Rightarrow \cancel{x = y}$ $\quad \Rightarrow -$

$$\Rightarrow \begin{bmatrix} 4x + 4y \\ -4x - 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x + 4y = 0 \qquad -4x - 4y = 0$

$\Rightarrow x + y = 0 \qquad x + y = 0$

Let us consider $\cancel{y = k}$ $x = k$ then $y = -k$

then Eigen vector is: $X = \begin{bmatrix} x \\ y \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Diagolization of matrices:

The transformation of a matrix into diagonal form is known as diagonalization.

formula of any ~~square~~ diagonalization of matrix: $D = P^{-1}AP$.

Q. Define the 2×2 matrix

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \end{bmatrix}$$

The inverse of P is $P^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

Sol. The similarity transformation

is $D = P^{-1}AP$

$$= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 + 2 & 1/2 + 0 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 1/2 \\ 3 & 0 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 0 + 3 & 0 \\ -\frac{3}{2}\cdot 2 + 3 & -2/2 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$